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## POST-WAR PLANNING IN MATHEMATICS

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Many teachers are nervously concerned over what may be the post-war status of school mathematics. The enormous expansion of the technical applications of the science under pressure of war has brought about a world-wide strengthening of mathematics in the school curriculum. Can this current academic primacy of mathematics be made permanent? Such is the question raised by those keenly mindful of the scant attention paid to this subject by the less recent curriculum makers.

A careful study of the matter should not discount the fact that in respect to mathematics, the war has served only to bring about greatly multiplied *uses* of mathematics a large proportion of which were already in existence. For, even in pre-war times there had been for many years a steadily growing public emphasis upon *applied* mathematics, rather than upon the logical or cultural aspects of the science.

In the light of this definite trend, a trend not rooted in any war, it could well be that the post-war school effort should first be directed to discovering the mathematical aids or needs of all the major peace-time industrial enterprises. Cooperative programs initiated between industry and the schools would then have sounder foundations. Who shall say that the cultures of mathematics would be impaired by being stemmed in its utilities?

S. T. SANDERS.

# On the Cevians of a Triangle

By N. A. COURT  
University of Oklahoma

1. The sides of a triangle  $(T) = ABC$  are met by the corresponding sides of the orthic triangle in three points lying on the same straight line.<sup>1</sup> This line is often called the *orthic axis* of  $(T)$ .

The orthic axis is also (a) the radical axis of the circumcircle  $(O)$  and the nine-point circle  $(N)$  of  $(T)$  (CG., p. 170, ex. 6) and (b) the trilinear polar of the orthocenter  $H$  of  $(T)$  (CG., p. 222, ex. 1)

To these properties may be added the following:

**Theorem.** *The orthic axis of a triangle  $(T)$  is the polar of the centroid with respect to the polar circle of  $(T)$ .*

The orthocenter  $H$  and the centroid  $G$  of  $(T)$  are the centers of similitude of the circumcircle  $(O)$  and the nine-point circle  $(N)$  of  $(T)$ ,<sup>2</sup> hence the circle  $(HG)$  having the segment  $HG$  for diameter (this circle is often referred to as the *orthocentroidal circle* of the triangle) is the circle of similitude of the circles  $(O)$ ,  $(N)$ , and is therefore coaxial with them,<sup>3</sup> hence the circle  $(G)$  having  $G$  for center and belonging to the coaxial pencil  $(H)$ ,  $(HG)$ ,  $(O)$ ,  $(N)$  is orthogonal to  $(H)$ . Thus the radical axis of  $(G)$  and  $(H)$  is the polar of the centroid  $G$  with respect to the circle  $(H)$ .

2. The side  $BC$  of  $(T)$  is the polar of  $A$  for the polar circle  $(H)$ , hence  $A$  is conjugate to every point of  $BC$ , and therefore a circle  $(AP)$  having for diameter a cevian  $AP$  of  $(T)$  is orthogonal to  $(H)$ .<sup>4</sup> Thus the radical axis  $p$  of the circles  $(H)$ ,  $(AP)$  is the polar of the mid-point  $P'$  of  $AP$  with respect to  $(H)$ .

The radical axis  $h$  of  $(H)$  and  $(O)$  is the orthic axis of  $(T)$ , i. e., the polar of the centroid  $G$  for  $(H)$  (Art. 1), hence the point  $ph$  is the pole, for  $(H)$ , of the line  $P'G$ , and, moreover, the point  $ph$  belongs to the radical axis of the circles  $(O)$ ,  $(AP)$ .<sup>5</sup> Hence the radical axis of  $(O)$ ,  $(AP)$  joins the point  $ph$  to the point  $A$  common to those two circles. Now  $A$  and  $ph$  are the poles, for  $(H)$ , of the lines  $BC$ ,  $P'G$ , hence the

<sup>1</sup>Nathan Altshiller-Court, *College Geometry*, p. 124, Art. 230. This book will be referred to as CG.

<sup>2</sup>CG, p. 161, ex. 3.

<sup>3</sup>CG, p. 217, Art. 429.

<sup>4</sup>CG, p. 153, Art. 296.

<sup>5</sup>CG, p. 170, Art. 337.

line  $A-ph$  is the polar of the point  $P_0 = (BC, P'G)$ . Applying Ceva's theorem to the transversal  $P'GP_0$  and the triangle  $APA'$ , where  $A'$  is the midpoint of  $BC$ , we find that  $P_0$  is the symmetric of  $P$  with respect to  $A'$ , i. e.,  $P_0$  is the *isotomic point* of  $P$  on the side  $BC$ . Thus: *The radical axis of the circumcircle of a triangle (T) and the circle having a cevian of (T) for diameter is the polar, with respect to the polar circle of (T), of the isotomic point of the foot of the cevian considered.*

2a. If the cevian  $AP$  (Art. 2) coincides with the median  $AA'$ , the points  $P, P_0$  both coincide with  $A'$ , and the proposition becomes: *The radical axis of the circumcircle of a triangle (T) and the circle having for diameter a median of (T) is the polar of the foot of the median considered with respect to the polar circle of (T).*

This proposition may be proved directly, as follows. The polar of the mid-point  $A'$  of  $BC$  with respect to the polar circle  $(H)$  of  $(T) = ABC$  passes through  $A$  and is perpendicular to the line  $A'H$ , hence the median  $AA'$  subtends a right angle at the foot  $K$  of this perpendicular on  $A'H$ , and therefore  $K$  lies on the circle  $(AA')$  having  $AA'$  for diameter.

Let the line  $HA'$  meet  $AO$  in  $L$ . The segment  $OA'$  is parallel to  $AH$ , hence  $O$  is the mid-point of  $AL$ , i. e.,  $L$  is the diametric opposite of  $A$  on the circumcircle  $(O)$ . Now the diameter  $AL$  of  $(O)$  subtends a right angle at  $K$ , hence  $K$  lies on  $(O)$ . Thus  $AK$  is the radical axis of  $(O)$  and  $(AA')$ .

3. Consider three cevians  $AP, BQ, CR$  of  $(T)$  having a point  $M$  in common. Let  $P_0, Q_0, R_0$  be the isotomic points of  $P, Q, R$  on the sides  $BC, CA, AB$  of  $(T)$ . Since  $AP, BQ, CR$  are concurrent, the cevians  $AP_0, BQ_0, CR_0$  also have a point, say,  $M_0$  in common, as is readily shown by Ceva's theorem. The point  $M_0$  is said to be the *isotomic of M for the triangle (T)*.

Now consider the three circles  $(AP), (BQ), (CR)$  having for diameters the cevians  $AP, BQ, CR$ . The point  $P_0$  is the pole with respect to  $(H)$  (Art. 2) of the radical axis of  $(O)$  and  $(AP)$ , hence the trace  $X$  of this axis on the line  $BC$  is the pole, for  $(H)$ , of the line joining  $P_0$  to the vertex  $A$ . Consequently the points  $M_0$  and  $X$  are conjugate with respect to  $(H)$ .

Likewise,  $M_0$  is conjugate, with respect to  $(H)$ , to the points  $Y, Z$  analogous to  $X$  and corresponding to the circles  $(BQ), (CR)$  respectively. Hence: a. *The three radical axes of the circumcircle of a triangle (T) with the three circles having for diameters three concurrent cevians of (T) meet the respective sides of (T) in three collinear points.*

b. *The line joining these three points is the polar, with respect to the polar circle of (T), of the isotomic point, for (T), of the point common to the given cevians.*

Part a. of this proposition is due to Paul D. Thomas.<sup>6</sup>

The centroid  $G$  of  $(T)$  is obviously its own isotomic point for  $(T)$ , hence when the medians of  $(T)$  are taken for the three concurrent cevians  $AP$ ,  $BQ$ ,  $CR$ , we have: a. *The three radical axes of the circumcircle of a triangle  $(T)$  with the three circles having for diameters the medians of  $(T)$  meet the respective sides of  $(T)$  in three collinear points.*

b. The line joining these three points is the orthic axis of  $(T)$  (Art. 1).

Part a. of this proposition is due to V. Thébault.<sup>7</sup>

The reader may find it interesting to prove this proposition directly, using Art. 2a.

4. *Converse theorem. If the three radical axes of the circumcircle of a triangle  $(T)$  with the three circles having for diameters three cevians issued from the three vertices of  $(T)$  meet the respective sides of  $(T)$  in three collinear points, the three cevians considered are concurrent.*

Indeed, if the points  $X$ ,  $Y$ ,  $Z$  are collinear, the cevians  $AP_0$ ,  $BQ_0$ ,  $CR_0$  have in common the pole  $M_0$  of the line  $XYZ$  with respect to the polar circle  $(H)$  of the given triangle  $(T)$ , hence the given cevians  $AP$ ,  $BQ$ ,  $CR$  have in common the isotomic  $M$  of  $M_0$  with respect to  $(T)$ .

5. *Application.* The points of contact  $X$ ,  $Y$ ,  $Z$  of the sides  $BC$ ,  $CA$ ,  $AB$  of a triangle  $(T)$  with the inscribed circle are the isotomic points of the points of contact  $X_1$ ,  $X_2$ ,  $Z_3$  of the same sides with the escribed circles relative to these sides,<sup>8</sup> and the two sets of corresponding cevians are known to be concurrent.<sup>9</sup> Thus (Art. 3): *The three radical axes of the circumcircle of a triangle with the circles having for diameters the cevians passing through the points of contact of the sides with the corresponding ex-circles meet the respective sides in three points lying on a straight line. This straight line is the polar of the incenter with respect to the polar circle of the triangle.*

6. The radical axis of the circles  $(O)$  and  $(AP)$  (Art. 2) is perpendicular to the line of centers  $OP'$  of the two circles, hence the propositions in articles 2 and 3 may also be stated as follows: a. *The perpendicular dropped from a vertex of a triangle  $(T)$  upon the line joining the circumcenter to the mid-point of a cevian passing through the vertex considered has for its pole, with respect to the polar circle of  $(T)$ , the isotomic of the foot of the cevian considered.*

<sup>6</sup> This MAGAZINE Vol. 16, 1942, pp. 306-307, q. 432.

<sup>7</sup> *American Mathematical Monthly*, Vol. 49, 1942, p. 63.

<sup>8</sup> CG, p. 75, Art. 120.

<sup>9</sup> CG, p. 129, Art. 239 and p. 130, Art. 242.

b. Given three concurrent cevians of a triangle  $(T)$ , the perpendiculars dropped from the vertices upon the lines joining the mid-points of the respective cevians to the circumcenter of  $(T)$  meet the corresponding sides of  $(T)$  in three collinear points. The line joining these points is the polar, with respect to the polar circle of  $(T)$ , of the isotomic point, with respect to  $(T)$ , of the point common to the three given cevians.

The propositions in articles 4 and 5 may be restated in a similar manner. The task is left to the reader.

7. Note. The circles  $(H)$ ,  $(O)$ ,  $(N)$  being coaxial, the point  $ph$  (Art. 2) also belongs to the radical axis of the circles  $(AP)$  and  $(N)$ . Moreover, the circles  $(AP)$  and  $(N)$  have in common the foot  $D$  of the altitude  $AD$  of  $(T)$ . The reader may be interested to discover properties analogous to those relative the circles  $(O)$  and  $(AP)$  considered in the preceding paragraphs.

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# Quadrangle bordé de triangles isoscelès semblables

By V. THÉBAULT\*  
Le Mans (France)

Dans un important Mémoire (*Association française pour l'Avancement des Sciences*, † 1891, pp. 53 à 66), COLLIGNON a donné relativement à la figure formée par un quadrangle bordé de carrés des propriétés intéressantes auxquelles nous en avons ajouté d'autres (A. F. A. S., Bruxelles 1932, pp. 78 à 81). La présente Note considère la configuration plus générale constituée par un quadrangle bordé extérieurement (ou intérieurement) de quatre triangles isoscèles semblables dont les angles à la base ont une valeur arbitraire donnée.

1. *Notations.* Soient un quadrangle convexe  $ABCD$ , ( $AB=a$ ,  $BC=b$ ,  $CD=c$ ,  $DA=d$ ,  $AC=e$ ,  $BD=f$ );  $A'$ ,  $B'$ ,  $C'$ ,  $D'$  les sommets des triangles isoscèles semblables, dont l'angle à la base est  $\alpha$ , ( $0 \leq \alpha \leq \pi/2$ ), construits extérieurement sur  $AB$ ,  $BC$ ,  $CD$ ,  $DA$ ;  $A''$ ,  $B''$ ,  $C''$ ,  $D''$  les sommets des triangles isoscèles égaux aux premiers et construits intérieurement sur  $AB$ ,  $BC$ ,  $CD$ ,  $DA$ ;  $M$  et  $N$  les milieux de  $AC$  et  $BD$ ;  $\omega$  l'angle aigu des diagonales  $AC$  et  $BD$ ;  $S_1$ ,  $S_2$ ,  $S_3$ ,  $S_4$  les aires des triangles  $ABC$ ,  $CDA$ ,  $BCD$ ,  $DAB$ ;  $S$  l'aire  $ABCD$ .

2. Les triangles  $AA'D''$  et  $ABD$  ayant un angle égal

$$A'AB + BAD'' = D''AD + D''AB$$

compris entre côtés proportionnels, sont semblables. Par suite,

$$(1) \quad A'D''/BD = AA'/AB = 1/2 \cos \alpha = C'B''/BD;$$

donc  $A'D'' = C'B''$ . On obtient de même  $A'B'' = C'D''$ , puis  $B'A'' = D'C''$ ,  $B'C'' = D'A''$ , de sorte que

$$(2) \quad \begin{aligned} A'B'' = C'D'' = B'A'' = D'C'' &= AC/2 \cos \alpha = e/2 \cos \alpha \\ A'D'' = C'B'' = B'C'' = D'A'' &= BD/2 \cos \alpha = f/2 \cos \alpha. \end{aligned}$$

\*V. Thébault is one of the foremost scholars in the field of Modern Geometry. His contributions have appeared in numerous periodicals in his native France as well as in many other countries. His writings date from the beginning of the present century.

An army officer during the first World War, he found time, in the trenches of France, to continue his scientific work. And now, driven out by the brutal invader from his home town, and a while even from his native land, Thébault, with admirable courage, has kept up his research, of which this article is an example.—N. A. COURT.

†Nous emploierous, dans la suite, l'abréviation. A. F. A. S.

Comme les triangles  $AA'D''$  et  $AA''D'$ , par exemple, sont égaux et que leurs côtés homologues  $AD''$  et  $AD'$  se coupent sous l'angle  $D'AD'' = 2\alpha$ , les côtés homologues des parallélogrammes  $A'B''C'D''$ ,  $B'C''D'A''$  sont égaux et se coupent sous le même angle  $2\alpha$ .

Le barycentre  $\Omega$  des points  $(A', B', C', D')$  et  $(A'', B'', C'', D'')$  coïncide avec celui des sommets  $(A, B, C, D)$  du quadrangle fondamental au milieu du segment rectiligne  $MN$ ; la droite  $G'G''$  des centres  $G', G''$  des parallélogrammes  $A'B''C'D''$ ,  $B'C''D'A''$  passe par le point  $\Omega$ , et  $G'\Omega = \Omega G''$ . Lorsque  $\alpha$  varie, les points  $A'$  et  $C'$ ,  $B'$  et  $D'$  décrivent des divisions semblables sur les médiatrices des côtés opposés  $AB$  et  $CD$ ,  $BC$  et  $DA$  du quadrangle; il en résulte que les milieux  $G'$  et  $G''$  des segments rectilignes  $A'C'$  et  $B'D'$  se déplacent sur une droite  $\Delta \equiv G'G''$  passant au milieu  $\Omega$  de  $MN$ . Or, lorsque  $\alpha = \pi/4$ , on sait que les points  $G'$  et  $G''$  sont les sommets d'un carré  $MG'NG''$ .<sup>\*</sup> Donc, dans le cas général, les droites  $MN$  et  $\Delta \equiv G'G''$  sont rectangulaires et  $MG'NG''$  est un losange d'angle  $2\alpha$ , de sorte que les milieux  $G'$  et  $G''$  des diagonales  $A'C'$  et  $B'D'$  sont les sommets de deux triangles isocèles semblables, d'angle de base  $\alpha$ , construits sur la droite  $MN$  joignant les milieux des diagonales  $AC$ ,  $BD$  du quadrangle fondamental.

3. Dans le triangle  $A'BB'$ , par exemple, on a

$$\overline{A'B'}^2 = \overline{BA'}^2 + \overline{BB'}^2 - 2\overline{BA'} \cdot \overline{BB'} \cos (B+2\alpha);$$

d'où successivement, en observant que  $BA' = a/2 \cos \alpha$ ,  $BB' = b/2 \cos \alpha$ ,  $2ab \cos B = a^2 + b^2 - e^2$ , et par analogie,

$$(3) \quad \overline{A'B'}^2 = [(a^2 + b^2)(1 - \cos 2\alpha) + e^2 \cos 2\alpha + 4S_1 \sin 2\alpha] / 4 \cos^2 \alpha$$

$$(4) \quad \overline{B'C'}^2 = [(b^2 + c^2)(1 - \cos 2\alpha) + f^2 \cos 2\alpha + 4S_3 \sin 2\alpha] / 4 \cos^2 \alpha$$

$$(5) \quad \overline{C'D'}^2 = [(c^2 + d^2)(1 - \cos 2\alpha) + e^2 \cos 2\alpha + 4S_2 \sin 2\alpha] / 4 \cos^2 \alpha$$

$$(6) \quad \overline{D'A'}^2 = [(d^2 + a^2)(1 - \cos 2\alpha) + f^2 \cos 2\alpha + 4S_4 \sin 2\alpha] / 4 \cos^2 \alpha.$$

Il en résulte que

$$(7) \quad \overline{A'B'}^2 + \overline{C'D'}^2 = [(a^2 + b^2 + c^2 + d^2)(1 - \cos 2\alpha) + 2e^2 \cos 2\alpha + 4S \sin 2\alpha] / 4 \cos^2 \alpha$$

$$(8) \quad \overline{B'C'}^2 + \overline{D'A'}^2 = [(a^2 + b^2 + c^2 + d^2)(1 - \cos 2\alpha) + 2f^2 \cos 2\alpha + 4S \sin 2\alpha] / 4 \cos^2 \alpha,$$

car

$$S_1 + S_3 = S_2 + S_4 = S.$$

Les expressions de  $\overline{A'B'}^2$ ,  $\overline{B'C'}^2$ ,  $\overline{C'D'}^2$ ,  $\overline{D'A'}^2$  se déduisent des précédentes en remplaçant le signe  $+$  par le signe  $-$  devant  $4S_1 \sin 2\alpha$ ,  $4S_2 \sin 2\alpha$ ,  $4S_3 \sin 2\alpha$ ,  $4S_4 \sin 2\alpha$ , de sorte que

<sup>\*</sup>cfr. Collignon, A. F. A. S., *loc cit.*, et V. Thébault, A. F. A. S., *loc cit.*

$$(9) \quad \overline{A''B''^2} + \overline{C''D''^2} = [(a^2 + b^2 + c^2 + d^2)(1 - \cos 2\alpha) + 2e^2 \cos 2\alpha - 4S \sin 2\alpha] / 4 \cos^2 \alpha$$

$$(10) \quad \overline{B''C''^2} + \overline{D''A''^2} = [(a^2 + b^2 + c^2 + d^2)(1 - \cos 2\alpha) + 2f^2 \cos 2\alpha - 4S \sin 2\alpha] / 4 \cos^2 \alpha.$$

Dès lors, si  $AC = e = f = BD$ , on a, à la fois,

$$\overline{A'B'^2} + \overline{C'D'^2} = \overline{B'C'^2} + \overline{D'A'^2} \quad \text{et} \quad \overline{A''B''^2} + \overline{C''D''^2} = \overline{B''C''^2} + \overline{D''A''^2},$$

et réciproquement, ce qui, en vertu d'une propriété connue permet d'énoncer la proposition suivante:

**Théorème.** Dans un quadrangle convexe  $ABCD$ , dont les diagonales  $AC$  et  $BD$  sont égales, les diagonales  $A'C'$  et  $B'D'$ ,  $A''C''$  et  $B''D''$  des quadrangles  $A'B'C'D'$  et  $A''B''C''D''$  dont les sommets coïncident avec ceux de quatre triangles isocèles semblables construits extérieurement et intérieurement sur les côtés, sont perpendiculaires, et réciproquement.

Lorsque  $\alpha = \pi/4$ ,  $\cos 2\alpha = 0$ , et on a, quel que soit le rapport des longueurs des diagonales  $AC$  et  $BD$  du quadrangle fondamental,

$$\overline{A'B'^2} + \overline{C'D'^2} = a^2 + b^2 + c^2 + d^2 + 4S = \overline{B'C'^2} + \overline{D'A'^2},$$

$$\text{et} \quad \overline{A''B''^2} + \overline{C''D''^2} = a^2 + b^2 + c^2 + d^2 - 4S = \overline{B''C''^2} + \overline{D''A''^2}.$$

On retrouve ainsi cette propriété classique.

**Théorème.** Dans un quadrangle convexe quelconque  $ABCD$ , les diagonales  $A'C'$  et  $B'D'$ ,  $A''C''$  et  $B''D''$  des quadrangles  $A'B'C'D'$  et  $A''B''C''D''$ , dont les sommets coïncident avec les centres des carrés construits extérieurement et intérieurement sur les côtés, sont perpendiculaires, et réciproquement.

Dans le triangle  $A'B''C'$ , on a

$$\overline{A'C'^2} = \overline{A'B''^2} + \overline{C'B''^2} - 2\overline{A'B''} \cdot \overline{C'B''} \cos A'B''C';$$

d'où, puisque

$$\text{angle } A'B''C' = \pi + (2\alpha - \omega),$$

et, en vertu de (2),

$$(11) \quad \overline{A'C'^2} = [e^2 + f^2 + 2ef \cos(2\alpha - \omega)] / 4 \cos^2 \alpha.$$

Dans le triangle  $B'C''D'$ , on a également,

$$\overline{B'D'^2} = \overline{B'C''^2} + \overline{D'C''^2} - 2\overline{B'C''} \cdot \overline{D'C''} \cos B'C''D',$$

c'est-à-dire que

$$(12) \quad \overline{B'D'^2} = [e^2 + f^2 - 2ef \cos(2\alpha + \omega)] / 4 \cos^2 \alpha,$$

en observant que angle  $B'C''D' = 2\alpha + \omega$ .

Par analogie, on obtient

$$(13) \quad \overline{A''C''}^2 = [e^2 + f^2 + 2ef \cos(2\alpha + \omega)] / 4 \cos^2 \alpha$$

$$(14) \quad \overline{B''D''}^2 = [e^2 + f^2 - 2ef \cos(2\alpha - \omega)] / 4 \cos^2 \alpha;$$

de sorte que

$$(15) \quad \overline{A'C'}^2 - \overline{B'D'}^2 = ef \cdot [\cos(2\alpha + \omega) + \cos(2\alpha - \omega)] / 2 \cos^2 \alpha \\ = \overline{A''C''}^2 - \overline{B''D''}^2.$$

$$\text{Si} \quad \cos(2\alpha + \omega) + \cos(2\alpha - \omega) = 0,$$

on a, à la fois,

$$A'C' = B'D' \quad \text{et} \quad A''C'' = B''D'',$$

et réciproquement. Cette condition est réalisée dans les deux hypothèses suivantes:

$$1^\circ) \quad \cos 2\alpha = 0 \quad \text{et} \quad \alpha = \pi/4;$$

le quadrangle  $ABCD$  est quelconque.

$$2^\circ) \quad \cos \omega = 0 \quad \text{et} \quad \omega = \pi/2;$$

le quadrangle  $ABCD$  est *orthodiagonal*. On a donc ces propositions:

**Théorème.** Dans un quadrangle quelconque  $ABCD$ , les diagonales  $A'C'$  et  $B'D'$  (ou  $A''C''$  et  $B''D''$ ) du quadrangle  $A'B'C'D'$  (ou  $A''B''C''D''$ ), dont les sommets coïncident avec les centres des carrés construits extérieurement (ou intérieurement) sur les côtés, sont égales. (Théorème comme).\*

**Théorème.** Dans un quadrangle *orthodiagonal*  $ABCD$ , les diagonales  $A'C'$  et  $B'D'$  (ou  $A''C''$  et  $B''D''$ ) du quadrangle  $A'B'C'D'$  (ou  $A''B''C''D''$ ), dont les sommets coïncident avec ceux de quatre triangles isocèles semblables construits extérieurement (ou intérieurement) sur les côtés, sont égales.

Enfin, si on a, à la fois,  $AC = BD$  et  $\omega = \pi/2$ , le quadrangle  $ABCD$  est un pseudo-carré.†

**Théorème.** Dans un pseudo-carré, le quadrangle dont les sommets coïncident avec ceux de quatre triangles isocèles semblables construits extérieurement (ou intérieurement) sur les côtés, est lui-même un pseudo-carré, et réciproquement.

\*On peut ajouter, en vertu d'un Théorème précédent, que les diagonales  $A'C'$  et  $B'D'$  sont perpendiculaires, ainsi que  $A''C''$  et  $B''D''$ .

†Expression due à J. Neuberg, A. F. A. S., 1893, *loc cit.*

Les aires des parallélogrammes  $A'B''C'D''$  et  $A''B'C'D'$  ont pour expressions:

$$(16) \quad A'B''C'D'' = -ef \sin(2\alpha - \omega) / 4 \cos^2 \alpha$$

$$(17) \quad A''B'C'D' = -ef \sin(2\alpha + \omega) / 4 \cos^2 \alpha.$$

Pour que ces aires soient égales, il faut et il suffit que

$$\sin(2\alpha - \omega) = \sin(2\alpha + \omega),$$

ce qui exige que l'on ait

$$\sin \omega = 0 \text{ (hypothèse à écarter), ou bien } \cos 2\alpha = 0,$$

soit  $\alpha = \pi/4$ . Dans ce cas, les parallélogrammes  $A'B''C'D''$  et  $A''B'C'D'$  sont égaux.\*

L'aire du parallélogramme  $A'B''C'D''$  s'annule lorsque

$$\sin(2\alpha - \omega) = 0, \text{ d'où } \operatorname{tg} 2\alpha = 4g\omega, \text{ ce qui exige que } \alpha = \omega/2, \text{ car } 0 \leq \alpha \leq \pi/2.$$

De même l'aire du parallélogramme  $A''B'C'D'$  devient nulle quand

$$\sin(2\alpha + \omega) = 0, \text{ c'est-à-dire si } \alpha = (\pi - \omega)/2.$$

Dans chacun de ces deux cas, les points  $(A', B'', C', D'')$  et  $(A'', B', C'', D')$  sont collinéaires, ce qui permet, en vertu de propriétés précédentes, d'énoncer cette proposition:

**Théorème.** Sur les côtés opposés  $AB$  et  $CD$ ,  $BE$  et  $DA$  d'un quadrangle  $ABCD$ , dont les diagonales  $AC$ ,  $BD$  se coupent en un point  $P$ , on construit extérieurement et intérieurement les triangles isocèles  $AA'B$ ,  $CC'D$  et  $BA''A$ ,  $DC''C$  d'angle au sommet  $APD = \omega$  et les triangles isocèles  $BB'C$ ,  $DD'A$  et  $CB''B$ ,  $AD''D$  d'angle au sommet  $BPA = \pi - \omega$ . Les points  $(A', B'', C', D'')$  sont collinéaires ainsi que les points  $(A'', B', C'', D')$ . Les droites obtenues passent au point de concours des diagonales  $AC$ ,  $BD$ , bissectant leurs angles, et les points  $B''$  et  $D''$ ,  $C''$  et  $A''$  sont symétriques par rapport aux milieux des segments rectilignes  $A'C'$  et  $B'D'$ .†

Des relations (3) à (6) et de leurs analogues pour le quadrangle  $A''B''C''D''$ , il résulte d'abord que

$$(18) \quad \overline{A'B''}^2 + \overline{B'C''}^2 + \overline{C'D''}^2 + \overline{D'A''}^2 \\ = [a^2 + b^2 + c^2 + d^2](1 - \cos 2\alpha) + (e^2 + f^2)\cos 2\alpha - 4S \sin 2\alpha] / 2 \cos^2 \alpha$$

\*cfr. V. Thébault, A. F. A. S., 1932, loc cit.

† Ce Théorème généralise une propriété du quadrangle orthodiamétral que nous avons donnée. A. F. A. S., 1932, loc cit. ((V. T.))

$$(19) \quad \overline{A''B''^2} + \overline{B''C''^2} + \overline{C''D''^2} + \overline{D''A''^2} \\ = [(a^2 + b^2 + c^2 + d^2)(1 - \cos 2\alpha) + (e^2 + f^2)\cos 2\alpha - 4S \sin 2\alpha] / 2 \cos^2 \alpha,$$

puis, que

$$(20) \quad \sum [(\overline{A'C'^2} + \overline{B'D'^2}) - \sum \overline{A'B'^2}] \\ = [e^2 + f^2 - (a^2 + b^2 + c^2 + d^2)] \cdot (1 - \cos 2\alpha) / 2 \cos^2 \alpha,$$

et enfin que

$$(21) \quad \overline{G'G''} = \overline{MN} \cdot \operatorname{tg} \alpha.$$

Si l'on construit extérieurement (ou intérieurement) sur les côtés  $A'B'$ ,  $B'C'$ ,  $C'D'$ ,  $D'A'$  du quadrangle  $A'B'C'D'$ , par exemple, des triangles isocèles semblables de sommets  $A_1$ ,  $B_1$ ,  $C_1$ ,  $D_1$  et d'angles à la base  $\alpha$ , puis sur les côtés  $A_1B_1$ ,  $B_1C_1$ ,  $C_1D_1$ ,  $D_1A_1$  du quadrangle  $A_1B_1C_1D_1$  des triangles isocèles de sommets  $A_2$ ,  $B_2$ ,  $C_2$ ,  $D_2$  semblables aux premiers, et ainsi de suite, on a

$$\overline{G'G''} = \overline{MN} \cdot \operatorname{tg} \alpha, \overline{G_1'G_1''} = \overline{MN} \cdot \operatorname{tg}^2 \alpha, \overline{G_2'G_2''} = \overline{MN} \cdot \operatorname{tg}^3 \alpha, \dots,$$

et les distances  $G'G''$ ,  $G_1'G_1''$ ,  $G_2'G_2''$ , ..., des milieux des diagonales des quadrangles  $A'B'C'D'$ ,  $A_1B_1C_1D_1$ ,  $A_2B_2C_2D_2$ , ..., sont les termes successifs d'une progression géométrique de raison  $\operatorname{tg} \alpha$ .

Il y a lieu d'observer que les points  $G'$  et  $G''$  affectés d'indices pairs (2, 4, ...) sont situés sur la droite  $\Delta \equiv G'G''$ , de part et d'autre du point  $\Omega$ , tandis que ceux qui sont affectés d'indices impairs (1, 3, ...) appartiennent à la droite indéfinie  $MN$  et sont séparés par le point  $\Omega$ .

Si  $\alpha = \pi/4$ , les points  $(A', B', C', D')$ ,  $(A_1, B_1, C_1, D_1)$ ,  $(A_2, B_2, C_2, D_2)$ , ..., sont les centres des carrés construits extérieurement sur les côtés des quadrangles  $ABCD$ ,  $A'B'C'D'$ ,  $A_1B_1C_1D_1$ , ...; dans cette hypothèse,

$$\overline{G'G''} = \overline{MN} = \overline{G_1'G_1''} = \overline{G_2'G_2''} = \dots$$

Les milieux des diagonales de tous les pseudo-carrés successifs  $A'B'C'D'$ ,  $A_1B_1C_1D_1$ ,  $A_2B_2C_2D_2$ , ..., qui dérivent ainsi d'un quadrangle quelconque  $ABCD$  coïncident alternativement avec les sommets  $G'$  et  $G''$ ,  $M$  et  $N$  du carré  $MG'NG''$ .

N. B. Ces propriétés s'appliquent également aux quadrangles  $A''B''C''D''$ ,  $A_1'B_1'C_1'D_1'$ ,  $A_2'B_2'C_2'D_2'$ , ..., provenant de la figure composée du quadrangle  $ABCD$  bordé intérieurement de triangles isocèles semblables.

4. On peut étendre à un polygone convexe  $(P) \equiv A_1A_2 \dots A_n$ , de  $n$  côtés, certaines des propriétés qui précèdent. Ainsi, on a cette proposition:

**Théorème.** *Loient  $A_1', A_2', \dots, A_n'$  les sommets des triangles isoscèles, d'angle de base  $\alpha$ , construits extérieurement (ou intérieurement) sur les côtés  $A_1A_2, A_2A_3, \dots, A_nA_1$  d'un polygone convexe  $(P) \equiv A_1A_2 \dots A_n$ . Les sommets des triangles isoscèles, d'angle de base  $\alpha$ , construits extérieurement (ou intérieurement) sur les côtés du polygone dont les sommets sont les milieux de toutes les diagonales du polygone  $(P)$  coïncident avec les milieux des diagonales du polygone  $(P') \equiv A_1'A_2' \dots A_n'$ .*

Il suffit, en effet, de grouper les sommets du polygone  $(P)$ , quatre par quatre, pour le décomposer en quadrangles dont les segments rectilignes joignant les milieux des deux diagonales sont chacun la base commune à deux triangles isoscèles semblables, d'angle de base  $\alpha$ , tels que les sommets sont les milieux des deux diagonales correspondantes du polygone  $(P')$ , ceci en vertu de 2.

Les cas particuliers où  $n=6$  et  $n=8$  sont particulièrement intéressants et donnent lieu à ces deux propriétés.

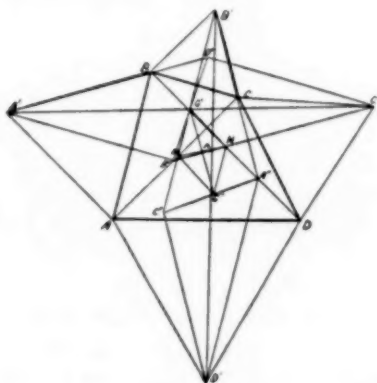
1°) Soient  $A_1', A_2', A_3', A_4', A_5', A_6'$  les centres des triangles équilatéraux construits extérieurement (ou intérieurement) sur les côtés  $A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_6, A_6A_1$  d'un hexagone  $(H)$ . Les milieux des diagonales principales  $A_1'A_4', A_2'A_5', A_3'A_6'$  de l'hexagone  $A_1'A_2'A_3'A_4'A_5'A_6'$  sont les sommets d'un triangle équilatéral.\*

\*V. Thébault, *Mathesis*, 1940, p. 96.

2°) Soient  $A_1', A_2', A_3', A_4', A_5', A_6', A_7', A_8'$  les centres des carrés construits extérieurement (ou intérieurement) sur les côtés  $A_1A_2, A_2A_3, \dots, A_8A_1$  d'un octogone  $(O)$ . Les milieux des diagonales principales  $A_1'A_5', A_2'A_6', A_3'A_7', A_4'A_8'$  de l'octogone

$$A_1'A_2'A_3'A_4'A_5'A_6'A_7'A_8'$$

sont les sommets d'un pseudo-carré.



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## The Influence of Tidal Theory Upon the Development of Mathematics

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It is now 300 years since Torricelli made his famous experiment which led to the invention of the barometer and to the tests with it on a mountain at the suggestion of Descartes and Pascal. Just 200 years have elapsed since Clairaut published his work on the Figure of the Earth in which the general equations of hydrostatics were formulated and since d'Alembert published his *Traité de Dynamique* in which numerous applications were made of the principle which bears his name. As Daniel Bernoulli had already published his researches in hydrodynamics and Clairaut was just beginning his work on the motion of the moon under the gravitational attraction of the sun and the earth, the time was ripe for the creation of a dynamical theory of the tides in place of the equilibrium theory in which the rotation of the earth is treated as very slow.

In 1744 d'Alembert made a first attempt to derive hydrodynamical equations with the aid of his principle and two years later he made a second attempt when he attacked the very difficult problem of the influence on the atmosphere of the changing positions of the sun and moon. He did not solve this problem but obtained some interesting results as by-products. Some of the difficulties of the problem may, perhaps, be understood when it is realized that at this time mathematicians were interested chiefly in the theory of *undamped* oscillations of a mechanical system. This theory began with the work of Brook Taylor on the vibrating string. Jean Bernoulli became interested in this work and inspired his pupils Daniel Bernoulli and Leonard Euler to work along similar lines. A system capable of free vibration was supposed to be disturbed by a transient force and left in a state of free vibration. After a series of disturbances the residual oscillation

naturally depends upon the past history. If the mechanical system has two natural periods of vibration which are nearly equal the residual oscillation may exhibit the phenomenon of beats, if the two periods are exactly equal the residual oscillation may be of increasing amplitude.\* If the past life of the system is regarded as infinite the residual oscillation may be built up from an infinite series of terms and there may or may not be convergence.

In the case of a simple jolt represented by  $f(t) = 1$  for  $0 < t < a$  and  $f(t) = 0$  for  $t < 0$  and  $t > a$ , the solutions of the equations

$$d^2u/dt^2 + m^2u = f(t)$$

$$d^4u/dt^4 + (m^2 + n^2)d^2u/dt^2 + m^2n^2u = f(t)$$

$$d^4u/dt^4 + 2m^2d^2u/dt^2 + m^4u = f(t)$$

are respectively  $m^2u = \cos m(t-a) - \cos mt$   $t > a$

$$u = \frac{\cos((nt) - \cos(mt)) - \cos n(t-a) + \cos m(t-a)}{n^2(n^2 - m^2)}$$

$$- \frac{\cos(mt) - \cos m(t-a)}{n^2m^2} \quad t > a$$

$$u = \frac{(t-a)\sin m(t-a) - t \sin mt}{2m^3} - \frac{\cos(mt) - \cos m(t-a)}{m^4} \quad t > a$$

In the case of a sustained oscillation in which the force acts over a long period the general solution consists of a particular integral and a complementary function which depends upon the past history of the system. About the middle of the eighteenth century we might imagine the question being asked. What shall we do about the past? The answer given by Laplace<sup>1</sup> in 1775 was essentially "Forget about it." Mathematically speaking Laplace's recipe was a theory of damped vibrations in which the free vibrations were supposed to be rapidly damped out by a kind of viscous friction with the result that the remaining vibration would have the same period as the exciting force when this consists of a term of type  $A \sin(nt+a)$ . The assumption of such a form seems at first sight to require a knowledge of the future of the force as well as its past but it should be noticed that when the solution of the equation

$$d^2u/dt^2 + 2kdu/dt + (k^2 + m^2)u = f(t)$$

\*It was not realized until the time of Weierstrass (1858) that the case does not actually occur in a correct theory of small undamped oscillations of a dynamical system. The account of the theory given by Lagrange is thus in one point incorrect.

is expressed in the form

$$mu = \int_{-\infty}^t e^{-k(t-s)} \sin m(t-s) f(s) ds$$

appropriate to the case in which  $u$  and  $du/dt$  are zero for  $t = -\infty$ , we can, with little error, replace  $-\infty$  by  $t-T$  where  $T$  is a large integral multiple of the period of the force  $f(s) = A \sin(n+a)$ . Making now the substitution  $s = t - T$  we get

$$mu = \int_0^T e^{-k(T-\tau)} \sin m(T-\tau) A \sin(nt+n\tau) d\tau$$

and this is of the form  $B \sin(nt) + C \cos(nt)$  or  $K \sin(nt+b)$ . Thus a vibration having the same period as the force is obtained by taking into consideration only the past and not the future of the force. The theory of damped vibrations is of great importance in the design of many scientific instruments in which the reading is to give a more or less faithful representation of the force. It was used by Laplace to show that in some cases at least, the equilibrium theory gives a good representation of the tide.

Laplace was not content, however, with a theory based on equations involving only one variable. In 1776 he started to calculate the particular integral having the same period as the force when the earth's rotation is taken into consideration. He assumed that the ocean covered the whole earth and had a constant depth. For simplicity he neglected the internal friction and looked for solutions of his partial differential equation that depended only on the latitude. If the equilibrium height of the tide at a place is  $H \sin^2 \theta \cdot \cos 2\phi$  or  $Hx^2 \cos 2\phi$  where  $\theta$  is the colatitude and  $\phi$  the hour angle of the disturbing body, Laplace replaced  $Hx^2$  by an unknown quantity  $L$  and looked for cases in which  $L$  depends only on  $x$ , thus obtaining the differential equation

$$(1-x^2)x^2 d^2L/dx^2 - xdL/dx + 2(x^2-4+2ex^4)L = -9Hx^2$$

To obtain a solution having the proper form at the poles he put

$$L = K_2 x^2 + K_4 x^4 + \dots$$

using first a method of approximation in which only a finite number of terms were used. Later in his *Mécanique Céleste* he used an infinite series and calculates the coefficients with the aid of the recurrence relation

$$2n(2n+6)K_{2n+4} - 2n(2n+3)K_{2n+2} + 4eK_{2n} = 0$$

Now  $K_2$  is known and  $K_4$  is at first sight arbitrary but it is necessary to make the series for  $L$  converge for  $x=1$  and so Laplace chooses  $K_4$  in such a way that the ratio  $K_{2n+2}/K_{2n}$  tends to zero as  $n$  tends to infinity. The expression for  $K_4/K_2$  then takes the form of an infinite continued fraction the word infinite being used here in the same way as in the term infinite series.

Some writers like Airy and Ferrel<sup>2</sup> were doubtful about the validity of Laplace's procedure but as Lamb points out<sup>3</sup>, many students read only the *Mécanique Céleste* which contains the finished product and ignored the original papers of Laplace in which more explanations are given. Lord Kelvin<sup>4</sup> became an enthusiastic advocate of Laplace's solution and showed where Airy was in error. Some misunderstanding has arisen by a slight confusion between the free vibrations having a period different from that of the force which Laplace eliminated by the introduction of damping and free vibrations having the period of the force which can only exist when the depth of the ocean has certain special values.

Laplace's method of using infinite continued fractions has been used for the solution of boundary problems associated with other linear differential equations such as Mathieu's equation and the equation defining the spheroidal wave functions. A slight modification of Laplace's method, ascribed by Forsyth<sup>5</sup> to Linstedt,<sup>6</sup> is used now to find the separation constants for the homogeneous equation and the associated periods of free vibration in many physical problems. Some uncertainty was felt about the convergence of the infinite continued fractions as may be judged, for instance, from the review of Linstedt's work in the *Jahrbuch der Fortschritte der Mathematik*. An attempt to discuss the convergence of the continued fractions used in the theory of the tides was made by K. Ogura<sup>2</sup> but, judging from the review in the *Jahrbuch*, his analysis requires modification. Lamb states<sup>3</sup> that the infinite continued fractions are convergent but does not give the test. A simple test which seems to meet the situation is provided by a modification of an old test which has been revived and made more definite by O. Szász<sup>8</sup> and W. Leighton.<sup>9</sup>

It is well known that if  $P_n/Q_n$  is the  $n$ th convergent of a continued fraction there are recurrence relations

$$P_n = b_n P_{n-1} + a_n P_{n-2}, \quad Q_n = b_n Q_{n-1} + a_n Q_{n-2}$$

where  $a_n$  is the numerator and  $b_n$  the denominator at the  $n$ th stage of the continued fraction. Now if  $P_n = b_n b_{n-1} \cdots b_1 p_n$ ,  $Q_n = b_n b_{n-1} \cdots b_1 q_n$  we have the equations  $P_n/Q_n = p_n/q_n$  and

$$p_n = p_{n-1} + c_n p_{n-2}, \quad q_n = q_{n-1} + c_n q_{n-2}$$

consequently the continued fraction is replaced by one in which the constituents  $b_n$  are all unity. For this type the test just mentioned states that it is sufficient for convergence that  $|c_n| \leq \frac{1}{4}$  and that this bound for  $|c_n|$  cannot be improved. The corresponding test for the convergence of the fraction in the original form is

$$|a_n/b_{n-1}b_n| \leq \frac{1}{4}$$

and this is the test that is applicable to the continued fractions which occur in tidal theory.

Laplace's theory has been developed by many writers such as Kelvin,<sup>4</sup> Rayleigh,<sup>10</sup> Darwin,<sup>11</sup> Lamb,<sup>12</sup> Hough<sup>13</sup> and Love<sup>14</sup>. Tables have been constructed by Doodson<sup>15</sup> and the theory has been applied to atmospheric oscillations by introducing an equivalent height for the atmosphere.<sup>16</sup> It is, perhaps, to the atmosphere that the theory is most directly applicable because the ocean does not cover the whole earth. The mathematical developments of tidal theory for the actual ocean consist partly of difficult analytical work for oceans with special boundaries such as meridians, parallels of latitude or ellipses and partly of simpler analysis applied to special regions in which there is much damping due to the water being shallow, the entrance to a basin being narrow, or to changes in level in the basin. Notable work has been done by French writers<sup>17</sup> who seem to like the methods of integral equations, by English writers like Proudman,<sup>18</sup> Doodson, Taylor, Jeffreys, Goldsbrough, Goldstein, Grace and Street who have done much numerical work in addition to elucidation of the theory and by writers of the Norwegian school whose work is summarized in the book of Bjerknes,<sup>19</sup> and his collaborators and in the paper of Solberg.<sup>20</sup> In the recent work of Proudman<sup>21</sup> a differential equation

$$d^2P/dx^2 + d^2P/dy^2 + e^{2x}(1 - w^2k^2)d^2P/dz^2 = 0$$

is obtained for the quantity  $P = p/\rho + V$ , where  $p$  denotes the pressure,  $\rho$  the density and  $V$  the potential of the gravitational and centrifugal force as in the work of Clairaut. The quantities  $e^x$ ,  $y$  and  $z$  are the usual cylindrical co-ordinates,  $w$  is the angular speed of the earth and  $\pi k$  the period of oscillation. Solutions are given for some special regions.

In the review of the paper by A. Weinstein<sup>22</sup> it is pointed out that the equation is similar to one considered by M. Brillouin and J. Coulomb<sup>23</sup> in 1933 and that according to J. Hadamard<sup>24</sup> the boundary conditions that are generally used are suitable for equations of elliptic type only while the partial differential equation is of hyperbolic type when  $w^2k^2 > 1$ . This is a question in which the present author<sup>25</sup> has been much interested in connection with the theory of an elastic fluid.

Notable work on equations which change in type has been done by M. Cibrario.<sup>26</sup>

A comparison of tidal theory with observation may be obtained by analysing hourly records of sea level or atmospheric pressure in such a way that a constituent with a given period is separated from other variations that seem random in comparison. If, for instance, the 10 o'clock readings are averaged over a large number of days variations with periods different from 12 hours will on the average contribute little to the mean and the result will be a variation with change of the hour of reading that has the period of 12 hours. This may be supposed to represent the solar tide. The period of the lunar semi-diurnal tide exceeds that of the solar tide by about  $1/28$  of its value and so a different method of averaging is needed to separate out the lunar tide.

At the suggestion of Laplace observations over a long period were carried out at Brest, a place which was very suitable for such work as there was much friction to reduce accidental free oscillations of the water.

Readings of atmospheric pressure in the tropics show a well marked semidiurnal variation of pressure. This type of variation is also a constituent of the variation of pressure in higher latitudes but is generally masked by the pressure changes associated with the weather. Analysis of the observations indicated that the solar semidiurnal variation is large in comparison with the lunar semidiurnal variation, a result which was rather unexpected because the lunar tide generating force is more than twice as large as the solar tide generating force and even then is too small to account for the observed variation on an equilibrium theory. Indeed, a surface over which the moon's gravitational potential has an assigned constant value, moves up and down at a place during the day by about 1 metre and this corresponds on an equilibrium theory to a pressure variation of about  $1/40$  of a millimetre.

To account for the magnitude of the semidiurnal variation of pressure Laplace suggested that it might be of thermal origin but Lord Kelvin<sup>27</sup> raised the objection that then one would expect there to be a well marked diurnal variation over the whole earth which is contrary to experience. As a matter of fact there are local diurnal variations of pressure as indicated by the land and sea breezes of coastal regions and the mountain and valley regional breezes of the Alps which have been ably discussed by E. Ekhart.<sup>28</sup> There is also a pressure variation which is less local and has a 8-hour period. This has a well marked seasonal change and is thought by Bartels<sup>29</sup> to be of thermal origin because a harmonic analysis of the intensity of solar radiation indicates that the term with a 8-hour period has a seasonal change of

just this type and a geographical distribution which, like that of the 6-hour pressure variation, can be represented roughly by means of a spherical harmonic of type  $p_4^3$ .

The existence of a 8-hour pressure variation was noted after J. Hann had made a long and careful study of the semidiurnal variation. According to Simpson,<sup>30</sup> this latter variation may be represented with some degree of accuracy by

$$p_2 = 0.937 \sin^3 \theta \cdot \sin(2l + 154^\circ) + 0.137(\cos^2 \theta - \frac{1}{3})\sin(2l - 2\phi + 105^\circ)$$

the unit being Imm. of mercury. The first term is regarded by Chapman<sup>31</sup> to be the result of both the sun's tidal and thermal action which are about equally effective. He was able to account for the phase of this wave on a resonance theory in which the magnification is about a hundred fold. The resonance theory was put forward by Lord Kelvin<sup>27</sup> in 1882 and was developed by M. Margules<sup>32</sup> who considered the oscillations of the atmosphere of a spheroidal earth. Chapman,<sup>33</sup> Pramanik and Topping have discussed the solar and lunar barometric pressure oscillations in the light of the resonance theory and have found that on this theory four types of pressure oscillations should be magnified by resonance. The theory of resonance is not so simple when the amplitude of vibration varies with position as it does in the atmosphere. A well known case which may be used for purposes of illustration is the case of a circular membrane. It was thought by Savart that a membrane would respond to almost any frequency above a certain limit but Bourget<sup>34</sup> found by a study of the roots of equations of type  $J_n(x) = 0$  that the possible frequencies seem to form an enumerable set. He surmised indeed that if  $m$  and  $n$  are positive integers the positive roots of  $J_m(x) = 0$  differ from those of  $J_n(x) = 0$  and from those of  $J_0(x) = 0$ , there being no common root of any of these equations. This has not yet been proved but it is known to be true for the first few values of  $m$  and  $n$ . The nodal circles and radii of the different type of normal modes of vibration seem also to be different in each case. When the membrane is in a state of forced vibration the distribution of energy amongst the different normal modes depends very much upon the distribution over the membrane of the exciting forces. In the case of a single force acting at a point of the membrane a particular mode will have no energy at all if one of its nodal lines passes through the point. There is in fact a kind of geometrical resonance in addition to that which depends upon a concordance of periods. This may be illustrated, perhaps, by taking the case of a vibration problem in which the forced vibration having the same period as the force is given by the solution of an integral equation

$$f(s) = F(s) - k \int_0^1 g(s, t) F(t) dt$$

in which  $k$  depends on the period of the force and  $f(s)$  upon its local distribution. To find the function  $F(s)$  which gives the local variation of displacement it is advantageous to expand  $f(s)$  in a series of the functions  $\psi_n(s)$  which satisfy the homogeneous equation

$$\psi_n(s) = k_n \int_0^1 g(s, t) \psi_n(t) dt$$

and correspond to the free vibrations of the system. If

$$f(s) = \sum c_n \psi_n(s)$$

and

$$F(s) = \sum C_n \psi_n(s)$$

we have the equation

$$\begin{aligned} \sum c_n \psi_n(s) \sum C_n \psi_n(s) - k \int_0^1 g(s, t) \sum C_n \psi_n(t) dt \\ = \sum C_n \psi_n(s) - \sum (k/k_n) C_n \psi_n(s). \end{aligned}$$

Hence

$$c_n = C_n(1 - k/k_n) \quad \text{or} \quad C_n k_n / (k_n - k).$$

The value of  $C_n$  may be large because  $k$  is nearly equal to  $k_n$  or it may be large because  $c_n$  is large, there are thus two kinds of resonance. In order that a force may excite a particular mode of free vibration or produce a vibration something like it the force must get a toe hold so to speak at a place where the vibration to be imitated has a concentration of energy.

This point is illustrated in the theory of atmospheric oscillations developed recently by C. L. Pekeris.<sup>35</sup> A distribution of temperature with height is found such that the atmosphere has a 12-hour oscillation such as is required for the resonance theory of pressure oscillations and also a  $10\frac{1}{2}$  hour oscillation of the type required to account by the theory of G. I. Taylor<sup>36</sup> for the velocity of propagation of the long waves produced by the eruption of Krakatoa. In the latter oscillation the energy is concentrated in the lower part of the atmosphere and there is a level at which the vertical velocity of the air is zero. In the 12 hour oscillation the amplitude and density of energy increase with height and there is a change of phase in the pressure oscillation such as is required by Chapman's dynamo theory of the diurnal variation of the earth's magnetic force.<sup>37</sup> On account of the

large damping by viscous forces at a level of 200 kilometers this type of oscillation requires a periodic force of permanent type for its maintenance and inequalities in it arising from accidental disturbances such as the eruption of Krakatoa may be expected to be rapidly eliminated.

Pekeris shows that the existence of a semidiurnal free oscillation in accordance with the resonance theory restricts in some measure the possible distributions of temperature in the upper atmosphere. The distribution adopted by Pekeris is one in which there is a constant lapse rate up to a height of about 10 km. then an isothermal region with an absolute temperature of about 240 up to a height of about 38 km. This in turn is followed by a rise in temperature with a gradient about equal in magnitude to the former lapse rate until a maximum temperature of about 350 K is attained at a height of about 62 km. A second lapse rate of the same magnitude brings the temperature back to its former constant value at a height of about 79 km. and the temperature subsequently remains at this value. The two gradients in the region 38-79 km. are slightly greater in magnitude than that near the ground. Pekeris has also made computations for some slightly different distributions of temperature.

In 1910 H. Lamb<sup>38</sup> discussed the oscillations (mainly horizontal) of an atmosphere with an assumed temperature distribution for the case of an earth without rotation. His differential equation for the divergence of velocity is of the second order and involves both the velocity of sound for the assumed distribution of temperature and also the quantities  $\sigma$  and  $k$  which occur in the factor  $e^{i\sigma t} J_0(kr)$  which gives the dependence on time and radius in the horizontal plane, the independent variable in the differential equation being the height above sea level. Lamb considers in particular the case of an atmosphere with uniform lapse rate and obtains a transcendental equation involving the confluent hypergeometric function which is hard to solve without suitable tables. In the case when  $\sigma^2/gk$  is very small,  $g$  being the acceleration of gravity, the equation is replaced by

$$\frac{1}{2}wJ_{n+1}(w) = \left( \frac{\beta}{\beta_1} - 1 \right) J_n(w)$$

where  $J_n(w)$  is the Bessel function of order  $n$ ,  $\beta$  gives the lapse rate of the actual temperature distribution and  $\beta_1$  gives the particular lapse rate in which the atmosphere is in convective equilibrium. Now this equation has been much studied as it occurs in the theory of vibration of circular plates, in the theory of the conduction of heat and in some electrical problems. A particular case of the equation was, I think, first obtained by Poisson.<sup>39</sup> The equation occurs in the

work of Moore<sup>40</sup> and Hobson<sup>41</sup> on expansions in series of Bessel functions. The equation has an infinite number of roots which are all real when the constants satisfy certain conditions. Ways of calculating the roots have been given by McMahon, Airey and others. For the large roots the following method may be used. Writing the equation in the form

$$wJ_{n+1}(w) = xJ_n(w)$$

where  $n$  is a known constant and  $x$  is a parameter whose value can be changed, we find from the recurrence relations for the Bessel functions that  $w$  satisfies the differential equation

$$(w^2 - 2nx + x^2)dw/dx = w.$$

We try to satisfy this by means of a series of type

$$w^2 = W^2 + C + C_1W^{-2} + C_2W^{-4} + C_3W^{-6} + \dots$$

where

$$W = \frac{1}{2}\pi(2n+1+4s),$$

$s$  is a large integer and  $C, C_1, C_2, \dots$  functions of  $x$  to be determined. In trying to find these we shall use the property that when  $x=0$  and  $2n=-1$  or  $-3$  the quantity  $W$  is always a root of the equation and so  $C, C_1, C_2, \dots$  should in this case be zero. A further fact which will be useful is that when  $x=2n$  the equation reduces to  $J_{n-1}(w)=0$  and so the formula should in this case agree with that obtained by putting  $x=0$  replacing  $n+1$  by  $n-1$  and  $s$  by  $s+1$ . Using primes to denote differentiations with regard to  $x$  we find that we have to satisfy the equation

$$\begin{aligned} (W^2 + C - 2nx + x^2 + C_1W^{-2} + C_2W^{-4} + \dots)(C' + C_1'W^{-2} + C_2'W^{-4} + \dots) \\ = 2W^2 + 2C + 2C_1W^{-2} + 2C_2W^{-4} + \dots \end{aligned}$$

Equating coefficients we find that  $C'=2$ ,  $C_1'=4nx-2x^2$ , etc. The first equations may be satisfied by writing

$$C = 2x - k(n + \frac{1}{2})(n + 3/2)$$

To find the value of the constant  $k$  we put  $x=2n$  and remark that in this case  $C$  should be equal to  $-k(n - \frac{1}{2})(n - 3/2)$ , consequently  $k=1$ . For convenience we now write  $C = 2x - m$  where  $m = (n + \frac{1}{2})(n + 3/2)$ . The differential equation for  $C_1$  may be satisfied by writing

$$C_1 = 2nx^2 - (2/3)x^3 + am + bm^2$$

and when  $x=2n$  we should have  $C_1 = aM + bM^2$  where  $M = (n - \frac{1}{2})(n - 3/2)$ . Hence we find that  $a = \frac{1}{2}$ ,  $b = -\frac{1}{3}$ . This process may be continued, the constant of integration in  $C_2$  being taken to be

$Am + Hm^2 + Km^3$  and when  $x = 2n$  the value of  $C_2$  should be  $AM + HM^2 + KM^3$ . The series found by McMahon for the case  $x = 0$  can, when squared, be used as a check. Another check may be obtained by putting  $n = -\frac{1}{2}$  and keeping  $x$  different from zero. The series for  $w$  can then be found by means of Lagrange's expansion using a method used successfully by Cauchy. Lamb's theory of atmospheric oscillations on a non-rotating earth has been used effectively by G. I. Taylor<sup>42</sup> who compares Lamb's differential equation with one obtained for the free oscillation of an atmosphere having the same temperature distribution upwards and enveloping the rotating earth. Taylor finds in fact that the limiting value of the velocity of propagation of waves of long period determines the periods of oscillation. The propagation of a pulse in the atmosphere has been studied by G. I. Taylor,<sup>42</sup> F. J. W. Whipple<sup>43</sup> and C. L. Pekeris,<sup>44</sup> the velocity of propagation is not greatly different from the ordinary velocity of propagation of sound.

In closing this discussion of the influence of tidal theory on the development of mathematics, reference should be made to the great advances in harmonic analysis and synthesis which have been made for the purpose of working up the observations and analyzing the tide producing forces. Many machines have been invented and a tide predicting machine was constructed by Lord Kelvin in 1876. Tests have been made to determine the accuracy and improvements in machines and methods of analysis have been made in recent years.<sup>45</sup> The literature of the subject is now enormous and so it cannot be summarized here.

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# *The Teachers' Department*

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---

## Need for Studies in Mathematical Education

By WILLIAM L. SCHAAF  
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1. *Introduction.* The immediate popularity of mathematical study is a happy circumstance which should not, however, be permitted to lull us into unwarranted complacency concerning the future. For it is at least problematical whether this wave of unprecedented general interest in mathematics will persist after the war. Current enthusiasm concerning aviation, navigation, meteorology, astronomy, maps and charts, ballistics, mechanics, and electronics may conceivably subside as rapidly as it sprang from nowhere. To assume naively that it will last beyond a point to which it will be carried by sheer momentum would be short-sighted indeed. In a sense, it is unfortunate that our entry into the war virtually coincided with the publication, late in 1940, of two outstanding reports, the one sponsored by the National Council of Teachers of Mathematics, the other by the Progressive Education Association. Presumably the full effect that they might have had on the course of mathematical education in this country will now never be known. Commendable as they are, unforeseen events have, for the moment at any rate, to some extent diverted interest from the former, and largely overshadowed the thesis of the latter.

It is the writer's belief that those of us who have faith in the worth of mathematical education would be wise to have some concern for the future prestige of the subject rather than remain smugly assured of the spontaneous continuance of that prestige. This admonition applies to college mathematics as well as to secondary mathematics. In fact, the considerations which prompt this conviction will doubtless have much to do with the policies of this Department under its new chairman. In view of the above, it is felt that research and investi-

gations in mathematical education should be assiduously pursued, *now*, even as technological research is being maintained and extended in industry with an eye to the future. There are many questions that need urgent study. It is the purpose of this paper to point out a few of the significant problems which demand attention.

2. *Adult Mathematical Education.* One of the most obvious areas (as yet barely touched) which has become particularly acute in the last few years is that of adult education in mathematics. In this area there arise a number of more or less related problems. Despite an abundance of literature on the subject of the psychology of the learning of mathematics, we still know very little about how the average adult learns (or attempts to learn) mathematics. Investigations ought to be made concerning the motives and interests of adults where mathematics is involved; of the difficulties encountered by adults; of the nature of language facilities; of the role of symbols, of imagery, of intuition, of accumulated adult experience, of aptitude, and of special ability and disabilities. On the side of methodology, there are to be considered not only adult learning of mathematics in conventional classes, but "home study", self-teaching materials, as well as learning by correspondence methods. Another fertile field of investigation, one that should have been pursued long ago, is a study of the public's viewpoint concerning mathematics—an evaluation analogous to Gruenberg's monograph on *Science and the Public Mind*, published in 1935. It is entirely possible, if we go about it the right way, that the momentary interest of large numbers of young men and women in mathematics can be made enduring, and will be capitalized by them through our guidance for other goals than those which loom large today.

3. *Significant Mathematical Concepts.* Turning for a moment to the field of secondary and junior college mathematics, it would seem evident that a number of challenging questions arise there also. One such group of questions to be investigated would deal with those concepts of mathematics which potentially contribute to greater general insights and broader appreciations. Which concepts are these? How difficult are they intrinsically? To what can they contribute? How are they learned? How should they best be taught? Some of them have to do with cultural considerations, particularly in connection with science and technology, as, for example, coordinate systems, rate of change, curve-fitting, nomograms, approximate computation, interpolation methods, probability, theory of errors, and so on, to mention but a few. Other significant concepts are more closely asso-

ciated with effective living, concepts fraught with socio-economic implications, such as functional relationships, graphic representation, ratios, rates, basic statistical concepts, exponential growth, index numbers, and the like. Finally, a group of concepts having to do with the mode of people's thinking, not unrelated, insofar as its significance is concerned, to politico-civic interests and activities. Sound thinking, resulting in effective action, is intimately associated with the concepts of assumption, hypothesis, definition, proposition, syllogistic methods, Euler's circles, relation, implication, inference, induction, deduction, etc.

Here again, psychological problems will be encountered, chiefly questions as to reciprocal learning (transfer of training), permanence of learning, problem-solving, individual differences, verbalism, functional learning, relation of mathematical "ability" to intelligence, special disability, evaluation of the outcomes of learning, etc. Although much has been written on these subjects, we are still in a state of confusion or comparative ignorance with regard to many of them.

4. *The Teacher's Background.* Considerable attention has been given in recent years, quite properly, to the training of mathematics teachers, particularly on the secondary and junior college level. Until the man-power shortage in the field of teaching became acute, the quality of the teaching personnel had been steadily improving. Most of the improvement, however, had to do with training in pure mathematics, on the one hand, and professional training (in the narrower sense) on the other. Comparatively little attention was paid to the teacher's total cultural background—that essential margin of learning over and above sheer mathematical learning and power, and in addition the methodological and administrative aspects of the teaching process. "Background", as here used, refers to related fields of human experience—the fine arts, the humanities, the practical arts, philosophy, history, language, science and technology. It is familiarity with these fields that engenders perspective and horizons which produce a resourceful, imaginative, stimulating teacher, and which assure more effective outcomes of the teaching process. The needed studies in this connection are concerned less with *what* these backgrounds should be than with *how* they shall be achieved. Can courses in pure mathematics contribute to their attainment? Can courses in educational theory augment them? Have the possibilities of "professionalized subject matter courses" been fully explored? Are "related courses" for teachers the solution? What can the literature of expository mathematics contribute? What activities can prospective teachers and teachers-in-service engage in to facilitate the acquisition of such cultural background? What is the role of museums, of mathematical

exhibits and collections, and of audio-visual aids in this respect? These are a few of the questions that might well be studied with profit.

A supplementary question also suggests itself. Is there a need for a mathematics teachers' handbook, similar to the handbooks for engineers and other professional workers? If so, what should such a handbook contain? a synopsis of appropriate mathematics? if so, on what level? Should it include discussions of methodology? of administration? of practical applications? of the history of mathematics? of expository mathematics? of the use of audio-visual aids? of extra-class activities—projects, field work, mathematical games, plays, contests? of mathematical literature? of professional literature? of mathematical recreations?

5. *Fuller Utilization of Existing Studies.* In conclusion, it remains to be pointed out that each year many studies are completed which do not receive as wide a dissemination as they deserve. We allude to the large number of masters' theses and doctoral dissertations that deal with one phase or another of mathematical education. In general, only the latter find their way into printed form. To be sure, many of the former are available for borrowing through an inter-library arrangement with the United States Office of Education. Yet this hardly makes for the fullest possible use of these studies.

The writer has compiled a classified bibliography of the theses which have appeared from 1928 to 1940 inclusive; it contains over 1500 entries, exclusive of those dealing with elementary school arithmetic. Most of them are listed in an annual bibliography published by the United States Office of Education, where some are accompanied by a brief abstract or comment. It is probably true that some of these studies, especially the masters' essays, may be of doubtful value in the sense that they contribute nothing new. Yet even a new approach to familiar material, or the aggregate trends and emphasis of the matter dealt with in these monographs, may prove of considerable worth if their content were more generally accessible. To suggest the possibilities of their professional utility, a few random titles are set forth below; their citation is not to be construed as an index of merit, but simply to suggest the range and variety of subjects treated.

*Amount of Mathematics Used in Leisure Time; Newspaper Survey of the Occurrence of Number; Relation of Mathematics to Civilization; Effects of Unfamiliar Words in Problem Solving; Causes of Aversion for Mathematics; Growth of Mathematical Concepts from the 4th to the 12th Grade; Educational Implications of Descartes' Synthesis of Mathematics and Philosophy; Relation between Degree of Introversion and Scholarship in Mathematics; Opinions of Life*

*Insurance Executives Concerning the Curriculum; Mathematics Needed for Intelligent Reading of Periodicals and Journals; What Measurements Do People Know, and Why?; Number and Spatial Ability in Mathematics and Intelligence; Socially Significant Historical Material in Mathematics; Teaching Secondary School Mathematics for Appreciation; Mathematical Education in the Light of a Social Philosophy; Comprehension of Mathematical Concept of Newspapers and Magazines; Creative Expression in Mathematics; Function of Mathematics in Adult Education; Mathematical Abilities Desired by Business Executives; Uses of Mathematics in General Reading.*

It would seem that if teachers generally were acquainted at least with the gist of the findings and opinions contained in such investigations and reports, the fraternity of mathematics teachers would be immensely enriched thereby. Furthermore, an examination of these studies would avoid duplication of the efforts of others, and serve as a starting point in new studies undertaken. The least that should be done is to make generally available a continuous summary digest of past and current theses, indexed and evaluated in a suitable manner. They will surely never be fully utilized if they lie buried on the library shelves of the institutions which sponsored them.

#### ANNOUNCEMENT

A memorial honoring Dr. James McGiffert will be published in an early issue of this journal. For nine years immediately prior to his death on June 18 of this year Dr. McGiffert was a member of the NATIONAL MATHEMATICS MAGAZINE Editorial Board.

# Spherical Trigonometry---An Emergency Course

By R. B. McCLENON  
*Grinnell College*

The return to our mathematical curricula of a course in spherical trigonometry offers us a real opportunity to effect a permanent enrichment of the educational program.

The extension of the formulas and processes of plane trigonometry to those of the surface of the sphere is in the first place a generalization that widens the mental horizon of the student, especially if he has never done much thinking about 3-dimensional geometry. Secondly, the renewal of acquaintance with the terrestrial globe, maps and their construction, and the elementary problems of navigation, are a revelation to many for whom geography remained a closed book since grammar school days. Lastly, the applications to the celestial sphere provide the incentive for some simple astronomical observations, which can hardly fail to be of interest, and which should have lasting value.

To begin with, the evident analogy between the triangle formulas for the plane and for the sphere is often not sufficiently emphasized in the ordinary text-book presentations. To illustrate:

## *I. The Sine Formulas.*

Plane trigonometry: 
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where  $R$  is the radius of the circumscribed circle.

Spherical trigonometry: 
$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = K^*$$

## *II. The Cosine Formulas.*

Plane trigonometry: 
$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Spherical trigonometry: 
$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

\*The Theorem of Menelaus is readily proved from these Sine Formulas, either for plane or spherical triangles.

### III. The Half-Angle Formulas.

Plane trigonometry:

$$\sin^2 \frac{A}{2} = \frac{(s-b)(s-c)}{bc} \quad \cos^2 \frac{A}{2} = \frac{s(s-a)}{bc}$$

$$\tan^2 \frac{A}{2} = \frac{(s-b)(s-c)}{s(s-a)} = \frac{r^2}{(s-a)^2}$$

Spherical trigonometry:

$$\sin^2 \frac{A}{2} = \frac{\sin(s-b)\sin(s-c)}{\sin b \sin c} \quad \cos^2 \frac{A}{2} = \frac{\sin s \sin(s-a)}{\sin b \sin c}$$

$$\tan^2 \frac{A}{2} = \frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)} = \frac{\tan^2 r}{\sin^2(s-a)}$$

$$\left( \text{where } s = \frac{a+b+c}{2} \text{ in both plane and spherical trigonometry} \right)$$

The striking parallelism thus evidenced suggests applying further analogous procedures:

By multiplying together the first two formulas of III we get for the plane triangle:

$$\sin^2 \frac{A}{2} \cos^2 \frac{A}{2} \left( = \frac{\sin^2 A}{4} \right) = \frac{s(s-a)(s-b)(s-c)}{b^2 c^2}$$

so that 
$$\frac{a}{\sin A} = \frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}} = 2R.$$

And for the spherical triangle,

$$\frac{\sin^2 A}{4} = \frac{\sin s \sin(s-a)\sin(s-b)\sin(s-c)}{\sin^2 b \sin^2 c}$$

so that 
$$\frac{\sin a}{\sin A} = \frac{\sin a \sin b \sin c}{2\sqrt{\sin s \sin(s-a)\sin(s-b)\sin(s-c)}}.$$

### IV. The Half-Side Formulas.

By using the second spherical formula under II, or by applying III to the polar triangle, we get:

$$\sin^2 \frac{a}{2} = \frac{-\cos S \cos(S-A)}{\sin B \sin C} \quad \cos^2 \frac{a}{2} = \frac{\cos(S-B)\cos(S-C)}{\sin B \sin C}$$

$$\begin{aligned}\tan^2 \frac{a}{2} &= \frac{-\cos S \cos(S-A)}{\cos(S-B)\cos(S-C)} \\ &= \cos^2(S-A) \cdot \frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)} = \cos^2(S-A)\tan^2 R\end{aligned}$$

where 
$$S = \frac{A+B+C}{2}$$

and  $R$  is the (spherical) radius of the circumscribed circle to the triangle, as shown in the next paragraph.

*V. Circumscribed, Inscribed, and Escribed Circles.*

(1) It is evident that if  $O$  is the pole of the circumscribed circle to  $ABC$ ,  $\angle OBM = S - A (= \angle OCM)$  where  $M$  is the mid-point of  $BC$ . Hence from the right triangle  $OBM$ ,

$$\cos(S-A) = \tan \frac{a}{2} \cot R$$

or 
$$\tan^2 R = \frac{\tan^2(a/2)}{\cos^2(S-A)}$$

which (by IV) gives

$$\tan^2 R = \frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}.$$

(2) Exactly as for the plane triangle, ( $T$  being the point of contact of the inscribed circle on  $AB$  or  $AC$ )  $AT = s - a$ , etc., so that

$$\tan \frac{A}{2} = \frac{\tan r}{\sin(s-a)}$$

which identifies  $\tan^2 r$  in III and gives

$$\tan^2 r = \frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}.$$

(3) Likewise for the escribed circles: for the one opposite  $A$ ,

$$\tan r_a = \tan \frac{A}{2} \sin s.$$

Hence (from III)

$$\tan r_a = \sqrt{\frac{\sin s \sin(s-b)\sin(s-c)}{\sin(s-a)}} = \frac{\sin s}{\sin(s-a)} \tan r.$$

And combining the three,

$$\tan r_a \tan r_b \tan r_c = \frac{\sin^2 s \tan^2 r}{\sin(s-a)\sin(s-b)\sin(s-c)} = \sin^2 s \tan r.$$

#### VI. The Tangent Formula and Napier's "Analogies".

A variation from the usual method of proving the Tangent Formula of plane trigonometry (by starting with the Sine Formula, or by a geometric proof) is this:

$$\text{From III,} \quad \frac{\tan(A/2)}{\tan(B/2)} = \frac{s-b}{s-a} = \frac{\sin(A/2) \cos(B/2)}{\cos(A/2) \sin(B/2)}$$

$$\text{whence} \quad \frac{\sin(A-B)/2}{\sin(A+B)/2} = \frac{(s-b)-(s-a)}{(s-b)+(s-a)} = \frac{a-b}{c}$$

$$\text{also} \quad \frac{\cos(A-B)/2}{\cos(A+B)/2} = \frac{s+(s-c)}{s-(s-c)} = \frac{a+b}{c} \quad (\text{"Mollweide's Formulas"})$$

$$\text{and by division,} \quad \frac{\tan(A-B)/2}{\tan(A+B)/2} = \frac{a-b}{a+b}.$$

For the spherical triangle,

$$\frac{\sin(A-B)/2}{\sin(A+B)/2} = - \frac{\sin(s-a) - \sin(s-b)}{\sin(s-a) + \sin(s-b)} = \frac{\tan(a-b)/2}{\tan(c/2)}$$

$$\frac{\cos(A-B)/2}{\cos(A+B)/2} = \frac{\sin s + \sin(s-c)}{\sin s - \sin(s-c)} = \frac{\tan(a+b)/2}{\tan(c/2)}$$

and from the polar triangle,

$$\frac{\sin(a-b)/2}{\sin(a+b)/2} = \frac{\tan(A-B)/2}{\cot(C/2)}, \quad \frac{\cos(a-b)/2}{\cos(a+b)/2} = \frac{\tan(A+B)/2}{\cot(C/2)}.$$

From either this pair or the preceding, we get by division the Tangent Formula:

$$\frac{\tan(A-B)/2}{\tan(A+B)/2} = \frac{\tan(a-b)/2}{\tan(a+b)/2}.$$

Geometric derivations of any or all of these and occasional historic references add variety and interest to the course.

#### VII. Applications.

Naturally, it is owing to its applications, especially in Navigation, that spherical trigonometry is being widely introduced or expanded to

meet the needs of pre-induction and pre-navy and pre-artillery training. And certainly the mathematical content of the course is greatly enriched by a genuine emphasis on problems involving terrestrial or celestial triangles, rather than just "spherical" triangles.

The problems of "great circle sailing," map and chart construction, determining position from celestial observations, to mention only a few, are of such general interest and value that, it is to be hoped, they will remain a part of the general education of the post-war generations. Spherical trigonometry deserves a place in our curriculum strictly on its merits and independent of emergencies.

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#### TEACHERS AND BIBLIOGRAPHIES

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A couple of years ago the present editor of this Department published a modest monograph containing upwards of 4000 bibliographic references to the periodical literature of mathematical education. Subsequent requests and correspondence would seem to indicate that teachers of mathematics are definitely interested in source materials, although it also appears that they regard such bibliographic material as most useful and valuable (1) when it relates to "content material" rather than to "methodology", and (2) when it covers specific or limited areas rather than broad or comprehensive fields. Presumably also, books and monographs ought to be included as well as periodical literature.

In casting about for ways and means of improving our service to both high school teachers and college teachers of mathematics, as well as to graduate students and prospective teachers, it occurs to us that our readers might find it helpful if this Department would, from time to time, publish brief, selected bibliographies on special subjects. We therefore ask that they express themselves on this matter. If those persons desiring such bibliographies will write to the editor, indicating the specific subject or topic of special interest, we shall be guided accordingly, and endeavor to accommodate their requests to the best of our ability and within the limits of our resources.—W. L. S.

## Brief Notes and Comments

Edited by  
H. A. SIMMONS

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6. *Relation Between the Radii of the Circumscribed and Inscribed Circles.* The relation  $R^2 - d^2 = 2rR$  connecting the circumradius  $R$  of the triangle  $ABC$ , the inradius  $r$ , and the distance  $d$  from the circumcenter  $O$  to the incenter  $I$  may be regarded as expressing the fact that the negative of the power of  $I$  with respect to the circle  $O$  is double the product of the distance  $r$  from  $I$  to the radical axis  $BC$  of the circle  $O$  and the circle through  $B$ ,  $I$ , and  $C$ , whose center  $K$  bisects the arc  $BC$ , by the line of centers  $OK$  which is equal to  $R$ . (The "power of a point with respect to a circle" is the difference between the square of the distance of the point to the center of the circle and the square of the radius of the circle.)\* Thus we can replace the usual proof of the relation  $R^2 - d^2 = 2rR$ , which is based on similar triangles, by an application of the theorem concerning the difference of the powers of  $I$  with respect to the circles  $K$  and  $O$ . (The difference of the powers of a point with regard to two non-concentric circles is twice the product of the distance between their centers by the distance from the point to the radical axis of the circles.)†

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J. H. BUTCHART.

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7. *Special Integration.* 1. We have

$$I = \int \frac{\sin x \, dx}{e^x + \cos x + \sin x} = \frac{1}{2} [x - \ln(e^x + \sin x + \cos x) + C],$$

as was shown in this MAGAZINE, Vol. 16, No. 7, April, 1942, pp. 355-356, Q. 446.

This integral was also considered in the *Boletín Matemático*, (Buenos Aires), Vol. 15, April, 1942, p. 35 by the editor of that journal,

\*Court, N. A. *College Geometry*, p. 163. Johnson Publishing Company, Richmond, Virginia, 1925.

†Johnson, R. A. *Modern Geometry*, p. 86. Houghton Mifflin Company, Boston, 1929.

Dr. B. I. Baidaff, who suggested the following elegant method of integration.

$$\text{If } J = \int \frac{(e^x + \cos x) dx}{e^x + \cos x + \sin x}$$

we have

$$I + J = x + C,$$

$$J - I = \ln(e^x + \cos x + \sin x) + C'.$$

Solving these two linear equations simultaneously we obtain for  $I$  the result given above, as well as the value

$$J = \frac{1}{2} [x + \ln(e^x + \cos x + \sin x)] + C'.$$

In the same issue of the *Boletin*, Dr. Baidaff proposes the following problem (No. 794). Evaluate

$$P = \int \frac{\cos x \, dx}{a \cos x + b \sin x}.$$

This integration may be performed in a manner similar to the above. Indeed, let

$$Q = \int \frac{\sin x \, dx}{a \cos x + b \sin x}.$$

We have

$$aP + bQ = \int dx = x + C$$

$$bP - aQ = \int \frac{d(a \cos x + b \sin x)}{a \cos x + b \sin x} = \ln(a \cos x + b \sin x) + C'$$

$$\text{hence } (a^2 + b^2)P = ax + b \ln(a \cos x + b \sin x) + C_1$$

$$(a^2 + b^2)Q = bx - a \ln(a \cos x + b \sin x) + C_2.$$

University of Oklahoma.

N. A. COURT.

# Problem Department

Edited by

E. P. STARKE and N. A. COURT

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to EMORY P. STARKE, Rutgers University, New Brunswick, N. J.

## SOLUTIONS

No. 483. Proposed by *Paul D. Thomas*, Sherburne, N. Y.

Given two non-concentric circles (*A*), (*B*). A third circle, (*C*), has a fixed radius but its center, *C*, moves on (*B*). Find the envelope of the radical axis of the circles (*A*), (*C*).

Solution by *R. C. Dragoo*, Norman, Oklahoma.

Let  $(a, 0)$ ,  $(0, 0)$ ,  $(h, k)$  be the cartesian coordinates of the centers *A*, *B*, *C* of the circles (*A*), (*B*), (*C*). The equations of these circles are

$$(x-a)^2 + y^2 = s^2, \quad x^2 + y^2 = r^2, \quad (x-h)^2 + (y-k)^2 = t^2,$$

and the equation of the radical axis of the circles (*A*), (*C*) is

$$2ax - 2hx - 2ky + h^2 + k^2 - t^2 + s^2 - a^2 = 0,$$

or, putting  $h = r \cos \theta$ ,  $k = r \sin \theta$ , and observing that  $h^2 + k^2 = r^2$ ,

$$(1) \quad ax - rx \cos \theta - ry \sin \theta + F = 0,$$

where *F* is a constant. Differentiating with respect to the parameter  $\theta$ , we have

$$(2) \quad rx \sin \theta - ry \cos \theta = 0.$$

Eliminating the parameter between (1) and (2) we obtain, as the equation of the required locus,

$$(r^2 - a^2)x^2 + r^2y^2 - 2aFx - F^2 = 0.$$

This conic is an ellipse, a parabola, or a hyperbola according as *r* is greater than, equal to, or smaller than *a*, i. e., when the center of

(A) lies inside, on, or outside the circle (B). The locus is a circle, when  $a=0$ , i. e., when the circles (A), (B) are concentric.

Also solved by the *Proposer*.

No. 485. Proposed by *Paul D. Thomas*, Sherburne, N. Y.

Prove that if the perpendiculars are drawn from the feet of the altitudes of a triangle to the adjacent sides, the feet of the perpendiculars lie on a circle.

Solution by *Walter B. Clarke*, San Jose, California.

Let  $H$  be the orthocenter of a triangle  $ABC$ ,  $H_a, H_b, H_c$  the feet of the altitudes; let  $B_a, C_a$  be the feet of the perpendiculars from  $H_a$  upon the sides  $AC, AB$ , and let  $C_b, A_b; A_c, B_c$  be the analogous pairs of points for  $H_b, H_c$ , respectively.

From similar right triangles we have

$$\frac{A_b C}{H_b C} = \frac{H_a C}{CA}, \quad \frac{B_a C}{H_a C} = \frac{H_b C}{BC}$$

hence 
$$\frac{A_b C}{B_a C} = \frac{BC}{AC}, \quad \text{i. e., } A_b B_a \parallel AB.$$

Again we have

$$\frac{CH_a}{H_a A_a} = \frac{CH}{HH_c} = \frac{CH_b}{H_b B_c}$$

hence  $A_c B_c \parallel H_a H_b$ . But the lines  $A_b C_a$  and  $AB$  are antiparallel for  $CB, CA$ , hence the same holds for the lines  $A_b B_a, A_c B_a$ , and therefore the four points  $A_b, B_a, A_c, B_c$  lie on a circle, say, (R).

Similarly, the points  $B_c, C_b, B_a, C_a$ , and  $C_a, A_c, C_b, A_b$  lie on circles, say, (P) and (Q).

Now the radical axes of the three circles (P), (Q), (R) taken in pairs are the sides of the triangle  $ABC$ , which is absurd, since the radical axes of three circles are concurrent. We conclude that the three circles are identical, which proves the proposition.

**BIBLIOGRAPHICAL NOTE.** This problem was first proposed by Eutaris (pen-name of Restiau, of the College Chaptal, Paris) in *Vuibert's Journal de Mathématiques élémentaires*, Vol. 2, 1877, pp. 30, 43, No. 60.

The same proposition, along with three others, is given in the book *Théorèmes et problèmes de géométrie élémentaire*, pp. 132-134, Paris, 1879, 6th edition, by Eugene Catalan (1814-1894).

The circle of the six points is known as the Taylor circle and is so referred to consistently, even by French authors, in honor of H. M.

Taylor (1842-1927) who studied this circle in an article entitled "A six-point circle connected with the triangle" in *Messenger of Mathematics* (London and Cambridge), Vol. XI (1882), pp. 177-179—N.A.C.

No. 489. Proposed by *D. L. MacKay*, Evander Childs High School, New York City.

Given the quadrilateral  $ABCD$ . Find two points  $X$  and  $Y$  on  $AB$  and  $CD$ , respectively, so that  $AX$ ,  $XY$ ,  $YC$  are proportional to  $m$ ,  $n$ ,  $p$ .

Solution by *J. E. La Fon*, University of Oklahoma.

A figure  $A'X'B'C'Y'D'$  similar to the required configuration may be constructed as follows. Draw  $A'X' = m$  parallel to  $AX$ ,  $X'Z' = p$  parallel to  $YC$ . With  $Z'$  as center describe a circle of radius  $n$ . Through  $A'$  draw a line parallel to  $AC$  meeting the circle in  $C'$ . (Thus there may be two, one or no solutions). Through  $C'$  draw a parallel to  $CB$  meeting  $A'X'$  in  $B'$  and a parallel to  $CD$  meeting parallel to  $AD$  through  $A'$  in  $D'$ . On  $C'D'$  take  $C'Y'$  equal to  $Z'X'$ . We now have  $A'B'C'D'$  similar to  $ABCD$  and  $A'X' = m$ ,  $X'Y' = n$ ,  $Y'C' = p$ . The required points  $X$ ,  $Y$  can now be constructed from the ratios

$$AX/m = YC/p = AC/A'C'.$$

No. 490. Proposed by *Howard D. Grossman*, New York City.

A chance event which can be either  $A$  or non- $A$  is repeated  $n$  times. Let  $E$  be the compound event of exactly  $r$  and no more  $A$ 's in succession. *Uspensky* (*Introduction to Mathematical Probability*, pp. 77-84) has given a treatment of the problem: find the probability that the sequence of  $n$  events will contain at least one  $E$ . The following problem is simpler and in many cases more significant: find the probable number of  $E$ 's and show that it is asymptotic to  $p'(1-p)^2n$ , where  $p$  is the probability of the occurrence of  $A$ .

Solution by the *Proposer*.

The concept of "probable number" of  $E$ 's may be arrived at as follows: Let  $k$  be a large number of trials and suppose that in each of  $k_0$  of these trials there were no  $E$ 's, in each of  $k_1$  there was one and only one  $E$ , in each of  $k_2$  there were just two  $E$ 's, etc., where  $k_0 + k_1 + k_2 + \dots = k$ . Then  $(k_1 + 2k_2 + 3k_3 + \dots)/k$  is the average number of  $E$ 's per trial and the limit of this fraction as  $n \rightarrow \infty$  is the probable number of  $E$ 's per trial. Since  $p_i = \lim_{n \rightarrow \infty} k_i/k$  is the probability that a trial will contain just  $i$   $E$ 's, we may write the probable number of  $E$ 's as  $p_1 + 2p_2 + 3p_3 + \dots$ .

In a sequence of  $n$  events there are  $n-r+1$  positions in which  $r$  successive  $A$ 's may be placed; one at the beginning, one at the end, and  $n-r-1$  intermediate. The probability that a sequence of  $n$  events shall commence with  $r$   $A$ 's followed by a non- $A$  is  $p^r(1-p)$ ; that it shall terminate with  $r$   $A$ 's preceded by a non- $A$  is  $p^r(1-p)$ ; that it shall contain in any other specified position  $r$   $A$ 's preceded by a non- $A$  and followed by a non- $A$  is  $p^r(1-p)^2$ . Now consider

$$(1) \quad 2p^r(1-p) + p^r(1-p)^2(n-r-1)$$

obtained by adding the probabilities. This is not the probability that a sequence of  $n$  events will contain at least one  $E$  (Uspensky's problem) because the separate probabilities added to obtain (1) are not mutually exclusive. But each of the probabilities has been counted in (1) precisely as many times as the corresponding sequence contains  $E$ 's. Hence (1) is the desired probable number of  $E$ 's. It is evidently asymptotic to  $p^r(1-p)^2n$ .

For an illustration, let the event be the tossing of a coin to obtain a head (whence  $p = \frac{1}{2}$ ), and let  $r=2$ ,  $n=15$ . Then the probable number of runs of two heads in 15 tosses is  $32768/32768=1$ , in agreement with (1), while the probability that the sequence of 15 tosses contain at least one run of two heads can be shown to be  $22025/32768$ . By this last sentence we mean of course: there are  $2^{15}=32768$  (equally likely) different ways in which a sequence of 15 tosses can result; if these were exhibited once each, we should find 32768 runs of two and only two successive heads contained in 22025 of the sequences. In fact there are 13370 of the  $2^{15}$  sequences which contain just one such run, 6762 containing two, 1692 containing three, 195 containing four, and 6 containing five. Thus we have

$$13370 + 6762 + 1692 + 195 + 6 = 22025,$$

$$13370 + 2(6762) + 3(1692) + 4(195) + 5(6) = 32768.$$

No. 491. Proposed by *Franklin Miller, Jr.*, Rutgers University.

A light stands at  $A$  within the angle formed by two intersecting mirrors. Show that the length of the path of a ray of light which returns to  $A$  after being reflected in both mirrors equals twice the distance between  $A$  and the intersection of the mirrors times the sine of the angle between them.

Solution by *D. L. MacKay*, Evander Childs High School, New York City.

The total path of the ray of light lies in a plane through  $A$  perpendicular to the intersection of the mirrors. In this plane let  $A'$

and  $A''$  be the symmetric of  $A$  with respect to the mirrors  $BC$  and  $BD$  respectively. If  $A'A''$  cuts  $BC$  in  $A_1$  and  $BD$  in  $A_2$ , then  $AA_1A_2A$ , the path of the ray of light, equals  $A'A''$ . Since  $A'B = AB = A''B$  and angle  $A'BA''$  is twice angle  $CBD$ , we have, from the isosceles triangle  $A'BA''$ ,  $\frac{1}{2}A'A'' = A'B \sin CBD$ , whence  $AA_1A_2A = A'A'' = 2AB \sin CBD$  as required.

EDITORIAL NOTE. A proof by *Nev. R. Mind* uses the fact that, by the laws of reflection, the mirrors are the external bisectors of the angles at  $A_1$  and  $A_2$  of the triangle  $A_1A_2A$ , whence  $B$  is the excenter of that triangle relative to the vertex  $A$ .

Also solved by *Paul D. Thomas, Richard K. Thomas, and Robert C. Yates*.

No. 492. Proposed by *Walter B. Clarke*, San Jose, California.

If  $\Delta$  is the area of the triangle  $ABC$ , show that

$$abc(a \cos A + b \cos B + c \cos C) = 8\Delta^2.$$

Solution by *J. Frank Arena*, Hardin, Ill.

Since  $A + B + C = \pi$ , we know that

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$$

$$\text{or } 2 \sin A \cos A + 2 \sin B \cos B + 2 \sin C \cos C = 4 \sin A \sin B \sin C$$

$$\text{or } \frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = 2.$$

The substitution of  $2\Delta/bc$ ,  $2\Delta/ca$ ,  $2\Delta/ab$  for  $\sin A$ ,  $\sin B$ ,  $\sin C$ , respectively, yields

$$\frac{a^2bc \cos A}{4\Delta^2} + \frac{ab^2c \cos B}{4\Delta^2} + \frac{abc^2 \cos C}{4\Delta^2} = 2,$$

which is equivalent to the required relation.

EDITORIAL NOTE. In his solution, *D. L. MacKay* notes that  $a \cos A + b \cos B + c \cos C$  is the perimeter of the orthic triangle (i. e. pedal triangle of the altitudes) of the triangle  $ABC$ . Noting also that  $8\Delta^2/abc$  is the product of the three altitudes, he gives the following expression to the proposed relation: In any triangle the product of the three altitudes equals the area times the perimeter of the orthic triangle. The *Proposer* calls attention to the connection with a theorem of Art. 157 of *N. A. Court, College Geometry*.

Also solved by *Albert Furman, William N. Huff, Lucille G. Meyer* and *Paul D. Thomas*.

No. 493. Proposed by *E. Hoff*.

If  $\Delta$  is the area of the triangle  $ABC$ , and  $R$  its circumradius, show that

$$a \cos^3 A + b \cos^3 B + c \cos^3 C = \Delta(1 - 4 \cos A \cos B \cos C)/R.$$

Solution by *D. L. MacKay*, Evander Childs High School, New York City.

Draw the altitudes  $AD$ ,  $BE$ ,  $CF$ . Since triangles  $AEF$ ,  $BDF$  and  $CDE$  are similar to triangle  $ABC$  with ratios of similitude  $\cos A$ ,  $\cos B$ ,  $\cos C$ , respectively, the area of triangles  $AEF$ ,  $BDF$ ,  $CDE$  are respectively equal to  $\Delta \cos^2 A$ ,  $\Delta \cos^2 B$ ,  $\Delta \cos^2 C$ . Hence

$$\Delta_1 = (1 - \cos^2 A - \cos^2 B - \cos^2 C),$$

where  $\Delta_1$  is the area of triangle  $DEF$ . By a familiar identity of trigonometry, this becomes

$$(1) \quad \Delta_1 = 2\Delta \cos A \cos B \cos C.$$

By the similar triangles noted above we have further

$$(2) \quad EF = a \cos A, \quad FD = b \cos B, \quad DE = c \cos C.$$

The formula of the preceding problem (No. 492) and the relation

$$(3) \quad abc = 4R\Delta \quad \text{give}$$

$$(4) \quad a \cos A + b \cos B + c \cos C = 2\Delta/R.$$

Analogous to (3) for the triangle  $DEF$  we have  $DE \cdot EF \cdot DF = 4R_1\Delta_1$ , where  $R_1$  is the circumradius of  $DEF$ . Thus using (2), (3) and (4) we find

$$(5) \quad R_1 = \frac{DE \cdot EF \cdot DF}{4\Delta_1} = \frac{abc \cos A \cos B \cos C}{8\Delta \cos A \cos B \cos C} = \frac{abc}{8\Delta} = \frac{R}{2}.$$

Draw the altitudes  $DD_1$ ,  $EE_1$ ,  $FF_1$  of triangle  $DEF$ . Since angles  $EDF$ ,  $FED$ ,  $DFE$  respectively equal  $180^\circ - 2A$ ,  $180^\circ - 2B$ ,  $180^\circ - 2C$  we have, analogous to (2),

$$E_1F_1 = a \cos A \cos 2A = a \cos A(2 \cos^2 A - 1),$$

$$F_1D_1 = b \cos B \cos 2B = b \cos B(2 \cos^2 B - 1),$$

$$D_1E_1 = c \cos C \cos 2C = c \cos C(2 \cos^2 C - 1).$$

Addition gives

$$2(a \cos^3 A + b \cos^3 B + c \cos^3 C) = (a \cos A + b \cos B + c \cos C) \\ - (F_1E_1 + F_1D_1 + D_1E_1).$$

The desired result now follows from (1), (4) and (5),

$$\begin{aligned} a \cos^2 A + b \cos^2 B + c \cos^2 C &= \Delta/R - \Delta_1/R_1 \\ &= \Delta(1 - 4 \cos A \cos B \cos C)/R. \end{aligned}$$

Also solved by *Albert Furman* and *J. Frank Arena*.

### PROPOSALS

No. 530. Proposed by *H. T. R. Aude*, Colgate University.

There exists an infinite number of sets of three integers or triads where each set may represent the sides of a triangle with only one angle equal to  $60^\circ$ . If for a given interval of the integers several such triads exist, then the one which corresponds to that triangle for which the angles are the closest to being equal we may call the "almost equiangular triangle" for that interval.

Under this assumption, remembering that but one angle is equal to  $60^\circ$ , find the almost equiangular triangle (a) for the interval 1 to 100; (b) for the interval 800 to 1600.

No. 531. Proposed by *Nev. R. Mind*.

If a circle has its center on an equilateral hyperbola and passes through the point of the hyperbola diametrically opposite to that point, then the remaining three points of intersection of the two curves form an equilateral triangle. A geometric proof is desired.

No. 532. Proposed by *E. P. Starke*, Rutgers University.

For the hyperbolic spiral  $\rho\theta = a$ , determine (1) the asymptote; (2) the point farthest below the polar axis and the point farthest to the left of the  $90^\circ$ -axis.

No. 533. Proposed by *V. Thébaud*, Le Mans, France.

If the Lemoine point of a triangle  $ABC$  lies on the circumcircle of the tangential triangle of  $ABC$ , we have

$$a^2 + b^2 + c^2 = 6R^2,$$

where  $a, b, c$  are the sides and  $R$  the circumradius of  $ABC$ .

No. 534. Proposed by *N. A. Court*, University of Oklahoma.

The line joining the center of the twelve point sphere of a tetrahedron ( $T$ ) to the midpoint of the segment determined by the Monge point and a vertex of ( $T$ ) passes through the corresponding vertex of the twin tetrahedron of ( $T$ ).

State the corresponding proposition for the orthocentric tetrahedron.

# Bibliography and Reviews

Edited by

H. A. SIMMONS and P. K. SMITH

Tables prepared under the direction of Arnold N. Lowan by the Federal Works Agency, Works Projects Administration for the City of New York and sponsored by the National Bureau of Standards:

1. *Tables of the Moment of Inertia and Section Modulus of Ordinary Angles, Channels, and Bulb Angles with Certain Plate Combinations.* 197+xiii. pages. 1941. \$2.00.

These tables are based on certain tables in the *Pocket Companion* published by the U. S. Steel Company. The computation of these particular tables was suggested by the Technical Division of the Bureau of Marine Inspection and Navigation of the Department of Commerce. Dr. L. J. Tuckerman of the Mechanics and Sound Division of the National Bureau of Standards was a consultant on this particular project.

2. *Tables of Sine and Cosine Integrals for Arguments from 10 to 100.* 185+xxxii pages. 1942. \$2.00.

This volume gives the values of

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt \quad \text{and} \quad \text{Ci}(x) = \int_{\infty}^x \frac{\cos t}{t} dt$$

correct to ten decimal places from  $x=10$  to  $x=100$  with  $\Delta x=0.01$ . The tables include second differences. Two volumes of previously published tables give the values of the same functions from  $x=0$  to  $x=10$  so that the present volume is really the third volume of a set.

This volume contains a Foreword by Professor J. A. Stratton of the Massachusetts Institute of Technology. The Introduction explains the method of computation of the tables, direct and inverse interpolation in the tables, and a description of the methods employed to check the manuscript. A Bibliography lists previous tables, texts, and articles that explain applications of these functions, and reference texts.

3. *Tables of Probability Functions.* Vol. I, 302+xxviii pages. 1941. \$2.00.

This volume consists of several tables for the function

$$H(x) = \frac{2}{\pi} \int_0^x e^{-\alpha^2} d\alpha$$

and its derivative. The tables give fifteen decimal-place accuracy for  $x$  from 0.0001 to 1.0000 with  $\Delta x=0.0001$  and fifteen decimal-place accuracy from 1.000 to 5.600 with  $\Delta x=0.001$ . A supplementary table gives the values of  $1-H(x)$  and  $H'(x)$  to eight significant figures from  $x=4.00$  to  $x=10.00$  with  $\Delta x=0.01$ .

Dr. T. C. Fry of the Bell Telephone Laboratories has written a very interesting Foreword for these tables. The Introduction contains an explanation of the methods of direct and inverse interpolation for use with these tables. It also explains the methods by which these tables were constructed. The preliminary manuscript contained

the entries to twenty decimal places and was subjected to a fifth difference test for accuracy. Additional difference tests were applied to the final tables to eliminate all chance of error.

4. *Miscellaneous Physical Tables*. 58+vi pages. 1941. \$1.50.

This volume contains tables for Planck's Radiation Functions and for several functions peculiar to electronics which were suggested by Professor A. E. Ruark.

5. *Tables of Natural Logarithms*. Vol. I: 501+xviii pages; Vol. II: 501+xviii pages; Vol. III: 501+xviii pages; Vol. IV: 506+xvii pages. 1941. \$2.00 per volume.

Volume I contains the natural logarithms of the integers from 1 to 50,000 each correct to sixteen decimals. Volume II contains the natural logarithms of the integers from 50,000 to 100,000 each correct to sixteen decimal places. Volume III contains the natural logarithms of the decimal numbers from 0.0001 to 5.0000 each correct to sixteen decimals and  $\Delta x = 0.0001$ . Volume IV contains the natural logarithms of the decimal numbers from 5.0000 to 10.0000 each correct to sixteen decimals and  $\Delta x = 0.0001$ .

Professor H. B. Dwight of the Massachusetts Institute of Technology has written a Foreword which is printed in each of the four volumes. He states: "It is very gratifying that the National Bureau of Standards has seen fit to sponsor a project which, by its impressive accomplishments in the short span of two years, bids fair to become the American counterpart of the Tables Committee of the British Association for the Advancement of Science, with a tradition of decades behind it."

The master sheets for these volumes were prepared by rounding off from a twenty decimal-place table that had been thoroughly checked by a differencing process. The master sheets were also checked by differences and by comparison in part with Thompson's *Logarithmica Britannica*.

These volumes are all printed by the photo-offset method and the possibility of errors is thereby practically eliminated. This reviewer quite agrees with Professor Dwight and congratulates the men behind this entire project.

North Carolina State College.

JOHN W. CELL.

*Business Mathematics*. By Cleon C. Richtmeyer and Judson W. Foust. McGraw-Hill Book Company, New York, 1943. xv+401 pages. \$2.75.

This is a revised edition of a book with the same title written by these authors and published in 1936. The present reviewer is not familiar with the earlier edition, but is unreservedly enthusiastic about the revised edition.

In the preface to the first edition the authors state: "This book is designed primarily for college classes that include students who are preparing to teach commercial arithmetic in secondary schools, students who are planning to enter various commercial fields, or students who wish a general course in practical mathematics. It is particularly adapted for use in teachers colleges, business colleges, and junior colleges. The text presupposes only the usual mathematics of the elementary and secondary schools". These objectives are met in a thoroughly satisfactory way by the revised edition.

Chapter titles are as follows: (I) Operations with Integers; (II) Common Fractions; (III) Decimal Fractions; (IV) Percentages and Applications; (V) Simple Interest and Bank Discount; (VI) Graphs; (VII) Measurement, Denominate Numbers, Geometric Figures; (VIII) Common Logarithms; (IX) Arithmetic Progressions and Short-

term Installment Buying; (X) Geometric Progressions and Compound Interest; (XI) Annuities and Applications; (XII) Equations; (XIII) An Introduction to Statistics. Adequate logarithm and investment tables are provided at the end of the text material.

Commendable features (but not necessarily in the following order of merit) which particularly impressed this reviewer are:

- (1) The lucidity, conciseness, and charm with which the exposition is written. Even in the hands of college freshmen, very little classroom time should be needed by the teacher to develop any of the topics, except perhaps in chapter XIII. Class sessions could thus be largely devoted to drill and discussion.
- (2) The astonishing wealth of *good* problems. These are excellently graded and wisely divided between formal drill exercises, which hit a topic from all angles, and intriguing thought problems of a practical character whose solution should really nail down the student's mastery of a topic. These "word" problems actually have the breath of life in them! Answers are given to odd numbered exercises and problems.
- (3) The fourteen sets of "Self-Test" problems given in addition to the regular exercises in each chapter, and as a general review at the end of the text. Answers are given to all of these problems. It is difficult to see how anyone but a hopeless dullard could resist matching his wits against these review problems.
- (4) The discussion, examples, and exercises in chapter III dealing with approximate numbers, significant digits, and computations with approximate numbers are excellent. Also in this chapter there is a clear exposition of contracted multiplication and division, a technique particularly useful in investment problems. Any teacher who has ever had to wrestle with a class over the sketchy presentation of this material in most of those texts where it is discussed at all will welcome the complete and sensible manner in which it is treated by this book.

Very little emphasis is placed on formal algebra, but this is in keeping with the spirit of the book, which aims to develop in the student the art of thinking numerically. There is enough good mathematical meat in the book to provide an invigorating diet at the college level!

On page 8, in the discussion of subtraction, it should be pointed out that 45 of the 10 subtraction combinations mentioned lead to negative numbers. The possibility of subtraction leading to a negative answer is not very clearly implied in the text presentation, although two of the nine parts in exercise 21 yield negative results implying that the authors at least had this possibility in mind.

The publishers have made the text easy reading from a mechanical point of view. Printing is clean, and stands out sharply on a good grade of gloss paper. Very few diagrams are required, but those that do appear are well presented.

*Northwestern University.*

M. E. WESCOTT.